Strong jump-traceability 2

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SJT and randomness
Random and c.e. sets

We focus on c.e. sets in light of the inherent enumerability of strong jump-traceability.

**Theorem (Diamondstone, Greenberg, Turetsky)**

*Every SJT set is computable from a c.e. SJT set.*
Random and c.e. sets

Thesis: c.e. sets and random sets have very little in common.

- An incomplete c.e. set cannot compute a random set [Arslanov].
- A sufficiently random set cannot compute a (incomputable) c.e. set [Hirschfeldt, Miller for exact bounds].

But not all is lost.
Kučera’s programme

**Theorem (Kučera)**
Every $\Delta^0_2$ Martin-Löf random set computes an incomputable c.e. set.

Q: what kind of random sets compute what kind of c.e. sets?

**Theorem (Hirschfeldt,Nies,Stephan)**
Every c.e. set computable from an incomplete ML-random set is *K-trivial*.

The converse is open (but wait for André’s talk).
The covering problem for SJT

**Theorem (Kučera, Nies; Greenberg, Turetsky)**

A c.e. degree is SJT if and only if it is computable from a Demuth random set.

The difference between Martin-Löf randomness and Demuth randomness is that when specifying components of the tests, we can change our minds a computably bounded number of times.
Lowness and randomness

We get a duality between a hierarchy of lowness notions of c.e. sets on the one hand, and of randomness (between Martin-Löf and weak 2-randomness) on the other.

- $K$-triviality corresponds to “Oberwolfach randomness” [Bienvenu, Greenberg, Kučera, Nies, Turetsky].
- Strong jump-traceability corresponds to Demuth randomness.

And the theme is that the strength of randomness is determined by what kind of approximations to the components of tests we allow.
What *many* random sets can compute

**Theorem (Greenberg, Hirschfeldt, Nies; DGT)**

The following are equivalent for a Turing degree $\alpha$:

- $\alpha$ is computable from every $\omega$-computably-approximable ML-random set;
- $\alpha$ is computable from every superlow ML-random set;
- $\alpha$ is SJT.

**Theorem (Greenberg, Hirschfeldt, Nies)**

The following are equivalent for a c.e. degree $\alpha$:

- $\alpha$ is computable from every superhigh ML-random set.
- $\alpha$ is SJT.

So paradoxically, in the context of randomness, both superlowness and superhighness are notions of strength.
If $A$ is SJT, then it is computable from every superlow random set.

Let $Y$ be a superlow random set.

- Start following Kučera. The use of the planned reduction $A \leq_T Y$ begin with the identity. If $Y \upharpoonright_k$ changes at stage $s$, declare that $Y \upharpoonright_k$ should computes $A \upharpoonright_s$. (Modifying this, we see that $Y$ can compute $A$ with "tiny use", [Franklin,Greenberg,Stephan,Wu].)

- We use little boxes to verify that $\alpha = A_s \upharpoonright_n$ is an initial segment of $A$. The "weight" of $\alpha$ is $2^{-|\gamma|}$, where $\gamma \prec Y_s$ is currently used for computing $\alpha$ from $Y$. Use $k$ boxes for strings of weight $2^{-k}$.

- If $\alpha \not\in A$, then we will have to enumerate $\gamma$ into a Solovay test. We need the total weight to be finite.
If $A$ is SJT, then it is computable from every superlow random set.

- Use metaboxes (one for each possible weight), to ensure that we believe an erroneous string with weight $2^{-k}$ at most $k$ times. Luckily, $\sum k2^{-k} < \infty$.

- When $Y|_k$ changes, we need to test a longer initial segment of $A$, and so need to run a new test - cannot use older boxes. If $Y$ is superlow then we can tell in advance how many parallel tests we may need at each weight, and so how many boxes we need for each metabox.

- The reason for defining the use as we did is so that the $k^{th}$ agent is only responsible for the latest version of $Y|_k$. The previous ones are passed over to $k - 1, k - 2, \ldots$. 
If $A$ is computable from every superlow random set, then $A$ is SJT.

A “phantom golden run construction”: we construct a random set which does not exist.

Suppose $A$ is computable from every superlow random set. We want to trace $J^A$.

- Start with a $\Pi^0_1$ class $P_0$ of randoms, and in the background, run the argument for the (super)low basis theorem: we get a sequence $P_0 = Q_0, Q_1, \ldots$, with $Q_i$ deciding the jump on the $i^{th}$ bit.
If $A$ is computable from every superlow random set, then $A$ is SJT.

- From this sequence, try to generate a trace for $J^A$. Given $n$ and a possible computation $J^A(n)$, with use $\alpha \prec A$, pick some $i$ (which depends on the required bound for the trace we are enumerating). Wait until $\alpha \prec \Phi_0(X)$ for every $X \in Q_i$, then believe. We will believe at most $2^i$ values (the number of possible versions of $Q_i$), hence the bound on the trace.

- While we wait, define $P_1$ to be the class of $X \in Q_i$ such that $\alpha \nless \Phi_0(X)$. Restart the process with $P_1$ and $\Phi_1$. Cancel when $A$ changes.

- If no level gives us a trace we let $\{Z\} = \bigcap P_n$. Then $Z$ does not compute $A$. We string together the superlow basis construction to show that $Z$ is superlow.
SJT and the c.e. degrees
First application: superlow cupping

**Theorem (Greenberg, Nies; DGT)**

Every SJT degree \( a \) is **superlow preserving**: for every superlow degree \( b \), \( a \lor b \) is also superlow.

**Corollary (Diamondstone)**

The notions of low cupping and superlow cupping differ in the c.e. degrees.
Lowness notions can often be partially relativised to obtain “weak reducibilities”. For example, $K$-triviality leads to $\leq_{LR}$, a relation which measures how well an oracle derandomises ML-random sets.

**Definition (Nies)**

Let $A, B \in 2^\omega$. Then $A \leq_{SJT} B$ if for every order function $h$, every $A$-partial computable function has a $B$-c.e. $h$-trace.

**Question**

Does $\leq_{SJT}$ imply $\leq_{LR}$?
**Definition**

- A set \( A \) is **LR-hard** if \( \emptyset' \leq_{LR} A \).
- A set \( A \) is **SJT-hard** if \( \emptyset' \leq_{SJT} A \).

**Theorem (Kjos-Hanssen, Miller, Solomon)**

A Turing degree is LR-hard if and only if it is almost everywhere dominating.

**Question (Nies, Shore, ...)**

In the c.e. degrees, is there a minimal pair of LR-hard degrees?
Pseudojump operators

There are direct constructions of incomplete LR-hard and SJT-hard c.e. degrees. An indirect approach uses pseudojump inversion.

**Definition (Jockusch, Shore)**

A **pseudojump operator** is a function $J : 2^\omega \rightarrow 2^\omega$ such that for all $A \in 2^\omega$, $J(A)$ is uniformly c.e. in $A$ and uniformly computes $A$. A pseudojump operator is increasing if for all $A$, $J(A) \geq_T A$.

**Theorem (Jockusch, Shore)**

For any pseudojump operator $J$ there is a c.e. set $A$ such that $J(A) \equiv_T \emptyset'$.

**Question (Jockusch, Shore)**

Can this be combined with upper-cone avoidance? Can one always invert to minimal pairs?

Partial answers by Downey, Jockusch, LaForte.
Restrictions on pseudojump inversion

**Theorem (Downey, Greenberg)**

There is no minimal pair of SJT-hard c.e. degrees. In fact, there is an incomputable c.e. set which is computable in every SJT-hard c.e. set.

**Corollary**

There is a natural, increasing pseudojump operator $J_{SJT}$ which cannot be inverted to a minimal pair, or while avoiding upper cones.
No minimal pair

This is an “inverted” box-promotion argument. Suppose that both $A_0$ and $A_1$ are c.e. and SJT-hard. We want to build an incomputable c.e. set $E$ below both $A_0$ and $A_1$.

- Friedberg-Muchnik actors will want to put a numbers into $E$. Such enumerations will require simultaneous permission from both $A_0$ and $A_1$.

- We can encourage $A_i$ to change by changing the values of a partial $\Sigma^0_2 = \Sigma^0_1(\emptyset')$ function $\psi$ and waiting for $A_i$ to enumerate the current value in its trace $T^{A_i}$ for $\psi$.

- If $z$ is a 1-box – the bound on the trace is 1 – then every change in $\psi(z)$ forces a change in $A_i$ below the use of enumerating the current value $\psi(z)$ in $T^{A_i}(z)$. We tie uses together so that this change permits a “follower” into $E$. 
No minimal pair

- If $z$ is a 2-box – we may need to ask twice before we get a change.
- But how do we get **simultaneous change** in $A_0$ and $A_1$? The only way is if we have 1-boxes on both sides.
- Boxes can be tied up by followers waiting to be realised. So the supply is limited.
- Box promotion is used to eventually manufacture 1-boxes from larger boxes. Again large metaboxes are used so that the gains from each promotion can be distributed to many mouths. See zig-zag picture.
Recall the question, “is there a minimal pair of c.e., LR-hard degrees?”. A solution may be found using the same technique.

- If every $K$-trivial degree is $\frac{1}{10} \log(n)$-jump traceable, then there is no minimal pair of LR-hard degrees.

(Recall that every $K$-trivial degree is $M \log n$-jump traceable for some $M$, but some $K$-trivial degree is not $o(\log n)$-jump traceable. So this is related to the problem of finding a combinatorial characterisation for $K$-triviality.)
The ideal SJTH♠ of all c.e. degrees which are reducible to all SJT-hard c.e. degrees is a new ideal in the c.e. degrees.

The extent of this ideal measures how restricted the construction of an incomplete SJT-hard c.e. set is.
Maximality

**Question**

*Is the ideal SJTH♠ principal?*

This question is difficult because the usual way for showing an ideal is not principal is by using... lower-cone avoidance.
An attempt at an answer

Theorem (Diamondstone, Downey, Greenberg, Turetsky)

SJTH♠ contains a superhigh set, but no SJT-hard set.

One hope is to use the superhighness hierarchy to obtain an answer.
Thank you