Relativization barriers, largeness measures for degree structures, and Martin's conjecture

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*Throughout the talk, I assume that "definable" sets of reals are determined. This follows from large cardinals.

Two fundamental properties of computation

Turing identified two fundamental properties of computation that are central to computability theory:

- We can effectively list all Turing machines.
- There is a universal Turing machine which can simulate any other Turing machine.

These two facts remain true for oracle Turing machines. This is what drives the phenomenon that essentially all proofs in computability theory relativize.

An example of relativization

Theorem (Turing, 1936)

The problem of determining whether a given Turing machine eventually halts is not computable.

Theorem (Relativized version)

For every oracle X, the problem of determining whether a given Turing machine eventually halts with X as an oracle is not computable from X.

We call the set of Turing machines which halt relative to X the **Turing jump** of X.

Martin's cone theorem

A **Turing cone** is a set of the form $\{Y : Y \geq_T X\}$ for some oracle X. That is, all oracles that can compute some fixed X. The **Turing degree** of an oracle X is all oracles equicomputable with X; the set of Y such that $X \geq_T Y$ and $Y \geq_T X$.

Theorem (Martin, 1969 . . .)

Any definable property of Turing degrees is either true on a Turing cone, or false on a Turing cone.

Roughly: Given any question in computability theory, above some point, the answer doesn't change when you relativize.

We will regard a set of degrees containing a Turing cone as "large".

A proof

Theorem (Martin, 1969 . . .)

Any definable property of Turing degrees is either true on a Turing cone, or false on a Turing cone.

Proof.

Play a game where two players alternate defining the bits of an oracle X. The first player wins if and only if the oracle has the given property.

$$\begin{array}{cccccc} \text{I} & x_0 & & x_2 & & \dots \\ \text{II} & & x_1 & & x_3 & \dots \end{array}$$

Work of Martin, Steel, and Woodin from the 70s and 80s implies that one of the players in such a game must have a winning strategy. If the first player has a winning strategy, the property must be true on a Turing cone. If the second player has a winning strategy, the property must be false on Turing cone. □

Martin's conjecture

Recall that if X and Y are Turing equivalent, then so are their Turing jumps X' and Y'. Hence, the Turing jump defines a map on the Turing degrees.

Conjecture (Martin, 1970s)

If f is a definable function on the Turing degrees that is not constant on a Turing cone, then f is equal to an iterate of the Turing jump on a Turing cone.

If we view mathematics through the lens of computability, the Turing jump has always played a fundamental role in how we organize and measure complexity. Martin's conjecture asserts that it is unique in this regard and is the only possible way to create a hierarchy of definability.

A theorem arising from partial results towards Martin's conjecture

Significant progress on Martin's conjecture has been made by Becker, Slaman, and Steel. One of the most interesting corollaries of their investigation is the following theorem:

Theorem (Slaman, 2000)

Suppose we have a definable reducibility $\geq_H$ coarser than Turing reducibility (i.e. some notion of hypercomputation) such that

- *If $X \geq_H Y$ and $X \geq_H Z$, then X can simultaneously hypercompute $X \geq_H Y \oplus Z$.*
- *If $X \geq_H Y$ and $Y \geq_T Z$, then $X \geq_H Z$.*

Then either $\geq_H$ is equal to Turing reducibility on a Turing cone, or $X \geq_H X'$ on a Turing cone.

If there were a natural definition of hypercomputation which didn't allow us to compute the halting problem, we would expect this fact to relativize to all oracles. However, Slaman's theorem proves this can not happen.

A contrast from computational complexity theory

Theorem (Baker, Gill, and Solovay, 1975)

For any oracle X, there are oracles $A \geq_{polytime} X$ and $B \geq_{polytime} X$ such that $\mathrm{P}^A = \mathrm{NP}^A$ and $\mathrm{P}^B \neq \mathrm{NP}^B$.

Of course, this implies that Martin's cone theorem is not true when Turing reducibility is replaced with polytime reducibility. The reason Martin's proof does not work in this context is that a strategy in a game takes an exponential amount of information to store!

Martin's proof tells us where to look for oracles witnessing the existence of a relativization barrier in complexity theory: they should be in the exponential hierarchy (under natural assumptions on how the relativization barrier itself relativizes).

An application to classification problems

Over the past 25 years, descriptive set theorists have developed a framework for comparing the difficulty of classification problems (technically, definable equivalence relations) in mathematics. This investigation has been remarkably successful both in proving practical results about problems of interest to working mathematicians and in understanding abstract features of the space of all classification problems.

In this setting, the classification difficulty of the polytime degrees is as high as possible–it is **universal**–identical to a number of problems from other areas of mathematics such as conformal equivalence of Riemann surfaces, isomorphism of finitely generated groups, and isomorphism of locally finite graphs.

An application to classification problems

One can use games similar to those of Martin's to define a notion of "largeness" for sets of polytime degrees (the large sets are simply no longer cones). This largeness notion and the universality of polytime equivalence drives the following theorem:

Theorem (M.)

If a universal countable Borel equivalence relation is partitioned into countably many Borel pieces, then one of these pieces is as hard to classify as the original problem.

This answered a question of Jackson, Kechris, and Louveau (2002).

Martin's conjecture has deep connections to the theory of classification problems. For example, Martin's conjecture would settle several questions about how classification difficulty is related to randomness. It would also imply drastic strengthenings of the above theorem.

Towards a solution to Martin's conjecture

Question

Suppose that Γ is a finitely generated group acting in a Borel way on $\mathbb{R}$, P is a countable set of infinite one-sided paths in the Cayley graph of Γ, and S is a countable set of sequences of natural numbers. Must there be a Borel function $f : \mathbb{R} \to \mathbb{N}$ such that if $(\alpha_0, \alpha_1, \ldots) \in P$ and $\{\alpha_0 \cdot x, \alpha_1 \cdot x, \ldots\}$ is infinite, then the sequence $f(\alpha_0 \cdot x), f(\alpha_1 \cdot x), \ldots$ is not in S.

Theorem (M.)

If the above question has a positive answer, then there is a counterexample to a slight generalization of Martin's conjecture: there is a Borel homomorphism from Turing equivalence to recursive isomorphism that is not uniform on any pointed perfect set.

More technical results imply that a negative answer to the above question would be significant progress towards Martin's conjecture.