<u>JMO mentoring scheme</u>

<u>Extra notes 2005-2006</u>

The notes provide supplementary ideas and theory to help mentees on the junior scheme. They are not meant to be complete, merely to provide a guide to topics which are not normally met in the school curriculum. Some older students may find a few useful ideas or approaches amongst topics with which they are largely familiar.

Section A - number theory

When tackling properties of integers, prime numbers, etc., a powerful method of analysing problems uses arithmetic modulo a given number. Thus the numbers {1, 11, 21, 31, 41, ...} all come in the same congruence class modulo 10 because when divided by 10, the remainder is always 1.

Further examples worked in modulo 7 : {4, 11, 18, 25, 32, ...} = 4 (mod 7)
{6, 13, 20, 27, 34, ...} = 6 (mod 7)

[Some texts use ≡ in this context, others use just = assuming that we know the context.]

Note that 4 + 6 = 3 (mod 7) because adding any number from the first class to any number from the second class gives a number in {3, 10, 17, 24, 31, ...} = 3 (mod 7). Proof :

$(4 + 7m) + (6 + 7n)$ where m, n integers

$= 10 + 7(m + n) = 3 + 7(m + n + 1)$ which is a member of class 3 since $m + n + 1$ is an integer.

Continuing to work in modulo 7 for ease of explanation, note that 0 is used to denote the class with 7 in it and not 7. Negative numbers are also included, e.g. −2 is in the class 5 (mod 7).

Generally : $K + 7n$ is in class K if the integer n is chosen so that $0 \le K < 7$.

- Example : if Guy Fawke’s Bonfire Night is on Saturday in 2005, what day is 22nd June 2005 ?
  30 − 22 (for rest of days in June) + 31 (Jul) + 31 (Aug) + 30 (Sep) + 31 (Oct) + 5 = 136 = 3 (mod 7)
  so 22nd June is 3 days earlier in the week, i.e. Wednesday.

Section B - Geometry

Many solvers of geometry problems like to draw decent diagrams using geometrical instruments, redrawing them as some ideas become irrelevant and other ideas take on increasing importance. Many of the questions use no more than you have probably met in mathematics classes - angle properties, Pythagoras, congruent triangles.

Triangles have many interesting properties, including those of their ‘centres’. The first is the centroid (the ‘balancing point’ of the triangle). There are 3 lines joining a vertex to the mid-point of the opposite side. These are the medians of a triangle. The three medians meet at the centroid (often shown as G) and split each others’ lengths in the ratio 2 : 1. Here is a possible proof using areas of triangles.

Let $|ABC|$ stand for the area of ΔABC. Let B^1 and C^1 be the midpoints of AC and AB respectively and let BB^1 and CC^1 meet in G : let AG meet BC at X. [We are going to prove that X and the midpoint of BC coincide.] Using common bases or common heights and because BB^1 is a median, it follows that

$|AB^1B| = |CB^1B|$ and $|AB^1G| = |CB^1G|$

∴ $|AGB| = |CGB|$

similarly $|CGA| = |CGB|$ because CC' is a median

∴ $|AGB| = |AGC|$

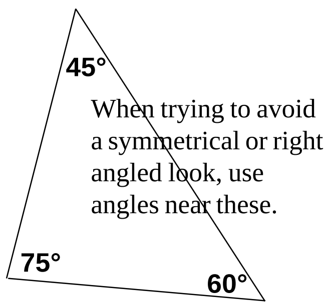
Now let $BX = k.CX$: By a similar argument, $|AGB| = k.|AGC|$

Hence k = 1 and X is the midpoint of BC.

Further : $|BGX| = |CGX|$

∴ $|BGX| = ½.|CGB| = ½.|ABG|$

∴ $GX = ½.AG$ • .



O : the circumcentre where the perpendicular bisectors of the three sides meet - from which a circle can be drawn through the three vertices of the triangle ;

H : the orthocentre where the altitudes of the triangle meet, i.e. the lines drawn from each vertex perpendicular to the opposite sides meet ;

I : the incentre where the angle bisectors at each vertex meet - from which a circle can be drawn to touch the sides of the triangle internally.

Section C - algebra

You may be familiar with multiplying out brackets for squares :

$(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$

Since a square number is always positive, the latter leads to : $a^2 - 2ab + b^2 \ge 0$.

This can be written in various ways, e.g. :

$a^2 + b^2 \ge 2ab$; ∴ $½(a^2 + b^2) \ge ab$; ∴ $½(a^2 + b^2) \ge \sqrt{(a^2b^2)}$

∴ $½(x + y) \ge \sqrt{(xy)}$ by replacing a^2 with x and b^2 with y .

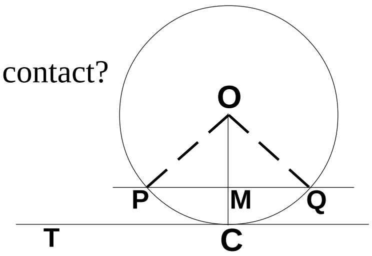
Section D - Greek alphabet

This is often used in mathematics when writers run out of names or wish to use a different style of letter. There are twenty-four characters usually written in this order :

Α	α	alpha	a	Ι	ι	iota	i	Ρ	ρ	rho	r
Β	β	beta	b	Κ	κ	kappa	k	Σ	σ	sigma	s
Γ	γ	gamma	g	Λ	λ	lambda	l	Τ	τ	tau	t
Δ	δ	delta	d	Μ	μ	mu	m	Υ	υ	upsilon	u
Ε	ε	epsilon	e	Ν	ν	nu	n	Φ	φ	phi	f
Ζ	ζ	zeta	z	Ξ	ξ	xi	x	Χ	χ	chi	c
Η	η	eta	h	Ο	ο	omicron	o	Ψ	ψ	psi	y
Θ	θ	theta	q	Π	π	pi	p	Ω	ω	omega	w

The first two columns show the upper and lower case Greek letters. The third is how we say them in the alphabet. The last is the key you use to access these on a computer keyboard using Symbol font.
Pronunciations vary. For example mathematicians tend to say ‘theeta’ for θ whereas classical Greek scholars say ‘thairta’ - similarly for ζ and η. ξ, ϕ, χ and ψ are prounounced to rhyme with π, i.e. ‘pie’.
A few Greek letters are common to Russian.
A number of English words have classical Greek origins, e.g. PSYCHOLOGY from ΨΥΧΗ (*spirit*) and ΛΟΓΟΣ (lots of meanings including *word*, *thought*, *reason*, etc.)
In mathematics α, β, γ are used for angles, coordinates (also the three types of nuclear particles/radiation).
θ, ϕ, ψ are commonly used for angles or coordinates and λ, μ, ν for coordinates.
Other common uses : Σ stands for *sum of* ; Π for *product of* ; ρ for *density* ; ω for *rate of turning* ; δ and ε for very small values ; Δ and δ for difference.

Section E - Circle Geometry

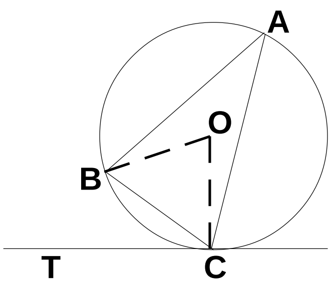


Why should a tangent be perpendicular to the radius of the circle at the point of contact?

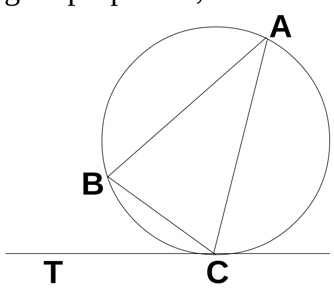
Let M be the midpoint of a chord PQ of the circle.
OP = OQ as radii of the circle $\therefore \angle OPQ = \angle OQP$.
Also PM = QM $\therefore$ ΔOPM is congruent to ΔOQM $\therefore \angle OMP = \angle OMQ$.
$\angle OMP + \angle OMQ = 180°$ $\therefore$ OM is perpendicular to PQ.
We can also prove the converse - if OM is perpendicular to PQ, then M is the midpoint of PQ.

Let OM meet circle at C. Construct TC parallel to PQ and $\therefore$ perpendicular to OC. Consider T to be where OP meets TC. Since OC > OM, then OT > OP (since ΔOPM is similar to ΔOTC) $\therefore$ T is outside the circle. This ensures that no point of the tangent TC lies *inside* the circle as the circle is a curve which is *convex* everywhere and that the tangent contacts the circle at just one point.

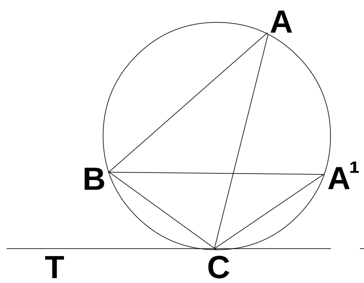
From radius, chord and tangent properties, we can deduce a number of theorems about angles in a circle.



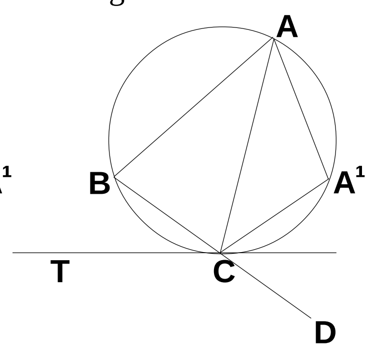
$\angle BOC = 2 \times \angle BCT$
$\angle BOC = 2 \times \angle BAC$



$\angle BCT = \angle BAC$



$\angle BAC = \angle BA^1C$
so $\angle ABA^1 = \angle ACA^1$
and two other pairs



$\angle A^1CD = \angle A^1AB$
$\angle A^1AB + \angle A^1CB = 180°$

For a more detailed account consult *Plane Euclidean Geometry : Theory and Problems* by A.D.Gardiner and C.J.Bradley published by the UK Mathematics Trust, School of Mathematics, University of Leeds, LS2 9JT.