IMOK OLYMPIAD

Notes for guidance

These notes are intended for pupils who have been invited to enter one of the Olympiad papers (Cayley, Hamilton or Maclaurin) from the Intermediate Mathematical Olympiad and Kangaroo (IMOK).

Each paper contains six questions, all requiring full written solutions.

Introduction

The questions on IMOK Olympiad papers are intended to be different. They will probably look unusual, so that it is not immediately obvious how to solve them, unlike a typical question from a school text-book. Furthermore, unlike an examination such as GCSE, there is no set syllabus content. However, the mathematics involved in the solutions should be familiar to most good pupils of your age, though the questions may appear unfamiliar.

You should have a copy of the 'official' solutions booklet to hand, but remember that this has been looked at by several people to produce a polished result. Not shown are the thought processes that went into finding these solutions, nor all the work that was done on scrap paper!

These notes are provided as a more informal guide, with the aim of showing you how it is possible to attack a difficult problem which may be unlike anything you have ever seen before.

What should you do to prepare yourself for an Olympiad paper? What will be needed above all is perseverance, to keep trying until some progress is made. There will be plenty of time to try the questions and keep worrying away at them until one of them yields up its secrets. It is certainly not a question of extra knowledge, more one of know-how.

When climbing a mountain, a strenuous effort may be needed to reach the top, to be replaced by a feeling of elation at the summit. In the same way, when solving a hard problem in mathematics there is nothing quite like the feeling one gets when the penny drops after a long struggle: "Ah, gotcha!"

Some general guidance is given below. On the following pages three sample Olympiad questions are discussed in detail, in an attempt to explain some of the thought processes you might go through when tackling them—these are *not* intended to be 'model' solutions!

Further reading

The UKMT web-site at `www.ukmt.org.uk` includes a list of resources, including useful books.

Strategy for the paper

Few candidates will do all the questions on the paper and doing three or four fully is really good. You should aim to solve one or two questions first, then if time allows go on to others. The early questions tend to be easier, the later ones harder, so it is fairly clear where to start!

Two hours gives plenty of time. Do not be afraid of spending time on one question and doing a lot of 'rough' work. But remember to allow time to write out your solution to each question in a clearly explained way. You will get few marks if you just hand in an 'answer', or a jumble of rough working.

Polished solutions like those in the official solutions booklet are not expected, though the examiners are always pleasantly surprised by the quality and ingenuity of some pupils' work. What you need to do is set out the mathematical steps in your solution in a clear and logical manner, so that the reader can follow your argument. Remember to include diagrams where necessary: it is better to have too many diagrams than too few.

Answers obtained by inexact methods will receive little credit. So, approaches like scale drawing should be avoided. Similarly, calculations which involve converting exact numbers such as π, $\sqrt{2}$ or even $\frac{1}{3}$ to approximate decimals like 3.14, 1.414 or 0.33 are also wasted effort. (One reason that calculators are forbidden is to help you avoid this.)

Cayley Paper—sample question

In this multiplication sum, p, q, r and s stand for different digits.

Find the digit which each letter represents, explaining how you know that you have found all possible solutions.

$$\begin{array}{rrr} & p & q \\ \times & r & q \\ \hline s & s & s \\ \hline \end{array}$$

It is clearly not a good idea in a question like this to start by trying to consider all values of the digits p, q and r—there are just too many cases to consider. So, we need to find a way of reducing the possibilities.

Standing back and looking at the overall pattern of the multiplication sum, we can make three potentially useful observations:

- two 2-digit numbers have been multiplied to give a 3-digit answer;
- the two 2-digit numbers end in the same digit q;
- all the digits in the answer are the same, namely s.

Is the first observation any help in reducing the possibilities? What size could the product of two 2-digit numbers be? Well, the smallest possibility is $10 \times 10 = 100$ and the largest is 99×99, which surely has more than 3 digits. Looking at possible products, the largest 3-digit number is 999, but there are a lot of ways to get a product close to this, from 10×99 to somewhere above 30×30. This line of attack does not seem helpful.

The second observation is more promising, since there are only ten cases to consider: q may take the values 0, 1, …, 9. What are the corresponding values for s?. Looking at the final digits in q^2, we see that s may be 0, 1, 4, 9, 6, or 5. But $q = 0$, 1, 5 and 6 are ruled out, because then q and s are the same. This looks encouraging, since we now have only the six cases $q = 2$, 3, 4, 7, 8 or 9 with corresponding answers for the multiplication of 444, 999, 666, 999, 444, 111. However, there is still quite a bit of work to do, since it is not clear how to find the values of p and r, or to be sure that we have found all possibilities.

For example, how do we find solutions to '$p\,2$' $\times$ '$r\,2$' $= 444$? One way is to work backwards and factorise 444. Now $444 = 2 \times 2 \times 111 = 2 \times 2 \times 3 \times 37$ and 37 is prime. It is now clear that the only way to write 444 as a product of two 2-digit numbers is 12×37, which is not of the form '$p\,2$' $\times$ '$r\,2$', so that we can eliminate this case.

The other cases may be considered in a similar way to complete a solution, but this work hints at a better approach using our third observation.

As we observed, the answer to the multiplication sum has three equal digits, so it is 111, 222, …, or 999. We want to know which of these numbers can be written as a product of two 2-digit numbers. So, as we did above with 444, it will be helpful to factorise them. But instead of doing them all separately, note that they are all of the form '$s\,s\,s$' $= s \times 111 = s \times 3 \times 37$ and 37 is prime. We could now consider each of the values of s in turn. Again there is a quicker approach, hinted at by the appearance of the 2-digit number 37 in the factorisation. This means that one of '$p\,q$' or '$r\,q$' is either 37 or a 2-digit multiple of 37, the only possibility being 74.

We are now in the luxurious position of having only two cases to deal with and the hard work is over. Consider the case 74. Then $q = 4$ and the sum is $?4 \times 74 =$ '$s\,s\,s$', so that $s = 6$ and '$s\,s\,s$' $= 666 = 9 \times 74$, which clearly does not work.

Finally, in the case 37, we have $q = 7$ and the sum is $?7 \times 37 =$ '$s\,s\,s$', so that $s = 9$ and '$s\,s\,s$' $= 999 = 27 \times 37$ and we therefore know the values of p and r, namely 2 and 3, either way round.

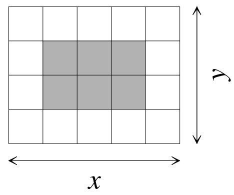
Hamilton Paper—sample question

A rectangular floor, which measures x feet by y feet, is covered in square tiles which are each 1 foot by 1 foot. All of the tiles on the perimeter of the rectangle are coloured blue, while all of the tiles on the interior of the rectangle are coloured yellow. There are three times as many yellow tiles on the floor as blue tiles.

(a) Show that x and y must satisfy the equation $(x-8)(y-8)=48$.

(b) Hence find all possible values for the area of the floor.

Reading the whole question, we see that we need to find the area of the floor and it is not obvious how to do so. However, the question setters have probably tried to help us by breaking the problem into two parts, so let us tackle (a) first.



It is clear that we need to find expressions for the numbers of blue and yellow tiles in terms of x and y and a schematic diagram is helpful. Here the yellow tiles are shaded grey and form a rectangle with dimensions $(x-2)$ feet by $(y-2)$ feet. Since each tile measures 1 foot by 1 foot, there are $(x-2)(y-2)$ yellow tiles. Careless counting of the blue tiles around the perimeter risks including the four corner tiles twice, or not at all. Counting carefully shows that there are $2x+2y-4$ blue tiles. We are told that there are three times as many yellow tiles as blue ones, so $(x-2)(y-2)=3(2x+2y-4)$.

Unfortunately, this is not the equation given in the question, so some algebraic rearrangement is going to be necessary. Expanding all the brackets gives $xy-2x-2y+4=6x+6y-12$ and therefore $xy-8x-8y+16=0$. We need 48 on the right-hand side, so add 48 to each side to get $xy-8x-8y+64=48$, which looks very promising, especially when we realise that the left-hand side factorises, to give $(x-8)(y-8)=48$, as required.

Before tackling (b), it is worth considering why the question asked for the rearranged equation in (a) rather than any other version. Could this lead to a method of solution? Standing back and considering the equation, we can interpret it as "multiply two numbers to get 48", which suggests a way to proceed. How do we find all the ways in which two numbers can multiply to give 48?

The most promising approach is to consider all the possible factorisations of 48, namely 1×48, 2×24, 3×16, 4×12 and 6×8. This looks encouraging, since we seem to have only five cases to deal with, but we need to remember that each of these may be written in reverse order, for example, 8×6. On further reflection, however, we see that we can avoid worrying about the order: there is nothing in the question to distinguish x and y, and we are only asked to find the area anyway. So, let us assume that x is the shorter length and hence $(x-8)$ will be less than $(y-8)$.

With this assumption, the table on the left shows all possible values for $(x-8)$ and $(y-8)$.

$x-8$	1	2	3	4	6
$y-8$	48	24	16	12	8

x	9	10	11	12	14
y	56	32	24	20	16

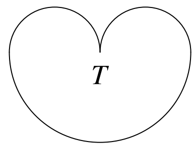
Adding 8 to each of these leads to the table of values for x and y on the right. Hence the possible values for the area, in square feet, are $9\times 56=504$, $10\times 32=320$, $11\times 24=264$, $12\times 20=240$ and $14\times 16=224$.

Have we found all possibilities and solved the problem? There is one cause for concern, namely negative numbers. It is clear from the context of the problem that x and y are positive numbers. However, it may be possible for both $(x-8)$ and $(y-8)$ to be negative, since all we know is that these terms multiply to give 48. Re-reading the method above shows that we assumed that these terms were positive.

Nevertheless, if we now make all the numbers in the first table negative and then add 8, none of the resulting pairs of values for x and y in the second table are both positive (the nearest being 2 and 0 in the last column). So there are no new solutions and our work above found all the possible areas.

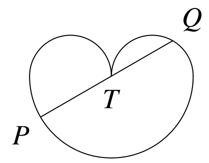
Maclaurin Paper—sample question

The figure shows three touching semicircles whose centres lie in a straight line. Two of the semicircles have radius 1 cm; these touch at T. The other semicircle has radius 2 cm.

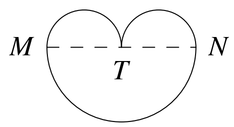


Prove that every straight line through T divides the perimeter of the figure into two parts of equal length.

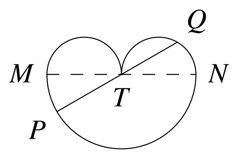
We need to draw the diagram and include a sample straight line PQ through T. The question asks us to prove that the line divides the perimeter into two equal parts. The obvious way to do this would be to find the lengths of the two parts of the perimeter. This looks a little complicated, so it is worth considering whether there is a better approach.



Whilst doing this, it may also be worth checking the special case when PQ lies along the diameters of the semicircles, shown as MN in the diagram. The part of the perimeter above MN has length equal to the circumference of a circle of radius 1 cm, which is 2π cm. The lower part has length equal to half the circumference of a circle of radius 2 cm, which is also 2π cm. So, as expected, the result is true for this special case.

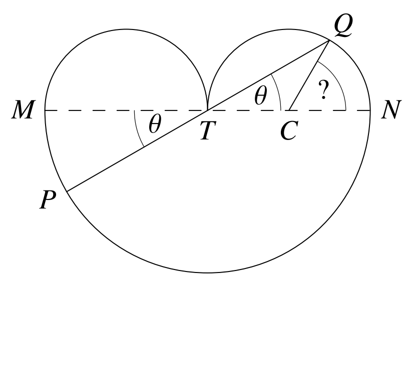


We clearly cannot prove the general result by considering all the special cases, but is there a way of relating the general result to the case MN? We know that the large semicircle forms half the perimeter in the diagram on the right. Consider rotating the line MN to the line PQ. This adds the arc NQ to the right side of the large semicircle and subtracts arc MP from the left side. To show that PQ cuts the perimeter in half, we therefore need to show that these two arcs, one added, the other subtracted, have equal length.



Not only have we simplified the problem, it is also now clear how to proceed: calculate the lengths of the two arcs. To do this we need to introduce an unknown quantity, since PQ is in an arbitrary position. How do we decide what to choose? Well, we are trying to find arc lengths and the formula for the arc length of a circle is $\frac{\theta}{360} \times 2\pi r$, where θ is the angle at the centre of the circle and r is the radius. That persuades us to let θ be the obvious angle at the centre of the large semicircle (the angle through which we rotated the line from MN to PQ).

Though this allows us to find the length of the arc MP, unfortunately it does not immediately help with arc NQ, since we do not know the angle at the centre C of the small semicircle, indicated ? in the diagram. However, remembering the theorem that 'the angle at the centre of a circle is twice the angle at the circumference', we deduce that the angle at C is 2θ. We now have all we need to find the arc lengths.



$$\text{Arc } MP = \frac{\theta}{360} \times 2\pi \times 2 \text{ and arc } NQ = \frac{2\theta}{360} \times 2\pi \times 1.$$

It is not necessary to simplify these arc lengths to see that they are equal. Hence we have proved the required result.