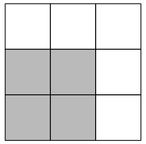
6. The numbers 1 to 9 are placed in the cells of a 3×3 square grid, one to each cell. In each of the four 2×2 blocks of adjacent cells, such as the one shaded, the four numbers have the same total T.

What is the maximum possible value of T?



Solution

The total of the numbers in the grid is $1 + 2 + \ldots + 9 = 45$.

Let the central number be x, let the four middle numbers along the sides be a, b, c, d and let the four corner numbers be p, q, r, s, as shown in the diagram:

p	a	q
b	x	c
r	d	s

Now add up the totals of the numbers in the four 2×2 blocks in two ways to show that

$$\begin{aligned} 4T &= 4x + 2(a + b + c + d) + (p + q + r + s) \\ &= 45 + 3x + (a + b + c + d). \end{aligned}$$

Since $x \leqslant 9$ and $x + a + b + c + d \leqslant 9 + 8 + 7 + 6 + 5 = 35$ we have $4T \leqslant 45 + 18 + 35 = 98$. Therefore $T \leqslant 24$.

However, it is possible to achieve the value 24 in several ways, for instance:

4	3	5
8	9	7
1	6	2

1	6	3
9	8	7
2	5	4

We deduce that the maximum value of T is certainly 24.

Note

Notice that, somewhat counter-intuitively, it is possible to achieve the maximum T using 8 in the central cell.

This booklet contains the questions and solutions for the follow-up competitions of the UKMT Intermediate Mathematical Challenge held in February 2011.

For the age ranges covered, see the details on the relevant paper.

As is usual, it is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Contents

EUROPEAN ‘KANGAROO’ MATHEMATICAL CHALLENGE
‘GREY’
Thursday 17th March 2011

Organised by the United Kingdom Mathematics Trust and the Association Kangourou Sans Frontières

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RULES AND GUIDELINES (to be read before starting):

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2. Time allowed: **1 hour**.
   No answers, or personal details, may be entered after the allowed hour is over.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. Candidates in England and Wales must be in School Year 9 or below.
   Candidates in Scotland must be in S2 or below.
   Candidates in Northern Ireland must be in School Year 10 or below.
5. **Use B or HB pencil only**. For each question mark *at most one* of the options A, B, C, D, E on the Answer Sheet. Do not mark more than one option.
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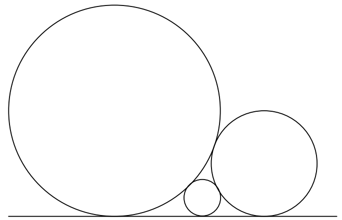
Maths Challenges Office,
School of Maths Satellite, University of Leeds, Leeds, LS2 9JT.
(Tel. 0113 343 2339)
http://www.ukmt.org.uk

We know that $x - 1$ is an integer. We deduce from equation 4.1 that $x - 1$ is a perfect square. Let $x - 1 = n^2$ for some non-negative integer n.

Now $0 < x < 2011$, so that $-1 < n^2 < 2010$ and hence $0 \leqslant n \leqslant 44$. But when $n = 0$ we have $x = 1$ and so $y = 0$, which is not allowed. Each other value of n gives a unique value of x, and therefore of y since $y > 0$.

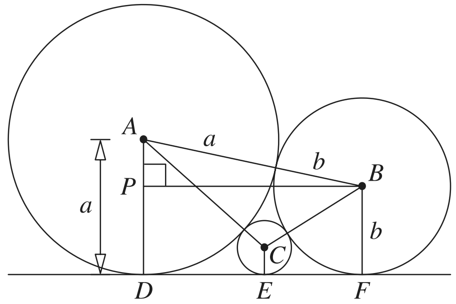
Hence there are 44 solutions to the given equation in positive integers.

5. Three circles touch the same straight line and touch each other, as shown.
   Prove that the radii a, b and c, where c is smallest, satisfy the equation
   $$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}.$$



Solution

We add some labels and lines to the diagram, as shown below: A, B and C are the centres; D, E and F are the points of contact of the circles and the straight line; and P is the foot of the perpendicular from B to AD.



Note first that AB, BC and CA, the lines of centres, pass through the points of tangency of the pairs of circles. Also, AD, CE and BF are perpendicular to the line DEF.

Then, by Pythagoras' theorem applied to triangle APB,
$$(a - b)^2 + PB^2 = (a + b)^2.$$

Therefore
$$PB^2 = (a + b)^2 - (a - b)^2$$
$$= 4ab.$$

Hence $DF = PB = 2\sqrt{ab}$. Similarly $DE = 2\sqrt{ca}$ and $EF = 2\sqrt{bc}$.

Since $DF = DE + EF$ we have
$$2\sqrt{ab} = 2\sqrt{ca} + 2\sqrt{bc}.$$

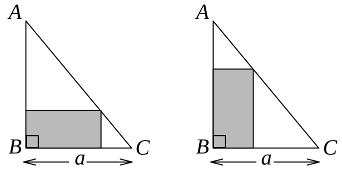
Dividing through by $2\sqrt{abc}$ gives the required result.

there are either 12 red socks or 32 red socks.

Notice that these two solutions are symmetrical; I must have 12 socks of one colour and 32 of the other.

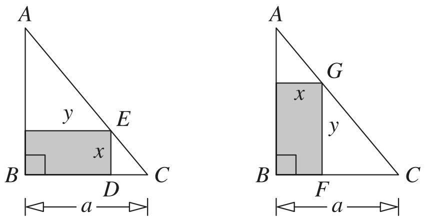
3. The diagrams show a rectangle that just fits inside right-angled triangle ABC in two different ways. One side of the triangle has length a.

Prove that the perimeter of the rectangle is $2a$.



Solution

We add some labels to the diagrams and let the dimensions of the rectangle be $x \times y$, as shown.



The triangles DCE and FCG are similar, so

$$\frac{x}{a - y} = \frac{y}{a - x},$$

which rearranges to give

$$x(a - x) = y(a - y)$$

that is,

$$a(x - y) = x^2 - y^2$$
$$= (x - y)(x + y).$$

Since $x - y \neq 0$ (because there are two *different* ways of fitting the rectangle) this implies that $a = x + y$. Thus the perimeter of the rectangle is $2x + 2y = 2a$.

Note

It follows that the triangle ABC is actually isosceles. Can you see why?

4. How many solutions are there to the equation $x^2 + y^2 = x^3$, where x and y are positive integers and x is less than 2011?

Solution

If $x^2 + y^2 = x^3$, then $y^2 = x^2(x - 1)$ and therefore, since $x > 0$,

$$\frac{y^2}{x^2} = x - 1. \qquad (4.1)$$

1. My broken calculator divides instead of multiplying and subtracts instead of adding. I type $12 \times 3 + 4 \times 2$. What answer does the calculator show?

A 2 B 6 C 12 D 28 E 38

2. A zebra crossing has alternate white and black stripes, each of width 50 cm. On a particular road, the crossing starts and ends with a white stripe and has 8 white stripes in all. What is the total width of this crossing?

A 7m B 7.5m C 8m D 8.5m E 9m

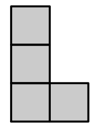
3. My digital watch has just changed to show the time 20:11. How many minutes later will it next show a time with the digits 0, 1, 1, 2 in some order?

A 40 B 45 C 50 D 55 E 60

4. In my street there are 17 houses. On the ‘even’ side, the houses are numbered 2, 4, 6, and so on. On the ‘odd’ side, the houses are numbered 1, 3, 5, and so on. I live in the last house on the even side, which is number 12. My cousin lives in the last house on the odd side. What is the number of my cousin's house?

A 5 B 7 C 13 D 17 E 21

5. The diagram on the right shows an L-shape made from four small squares. Ria wants to add an extra small square in order to form a shape with a line of symmetry. In how many different ways can she do this?



A 1 B 2 C 3 D 4 E 5

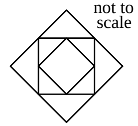
6. Felix the Cat caught 12 fish in 3 days. Each day after the first, he caught more fish than the previous day. On the third day, he caught fewer fish than on the first two days combined. How many fish did Felix catch on the third day?

A 5 B 6 C 7 D 8 E 9

7. Mary lists every 3-digit number whose digits add up to 8. What is the sum of the largest and smallest numbers in Mary's list?

A 707 B 907 C 916 D 1000 E 1001

8. The diagram shows three squares. The medium square is formed by joining the midpoints of the sides of the large square. The small square is formed by joining the midpoints of the sides of the medium square. The area of the small square is 6 cm^2. What is the difference between the area of the medium square and the area of the large square?



not to scale

A 3 cm^2 B 6 cm^2 C 9 cm^2 D 12 cm^2 E 15 cm^2

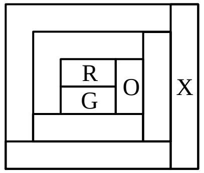
9. What is the value of $\dfrac{2011 \times 2.011}{201.1 \times 20.11}$?

A 0.01 B 0.1 C 1 D 10 E 100

10. Maria has nine pearls that weigh 1, 2, 3, 4, 5, 6, 7, 8 and 9 grams. She makes four rings, using two pearls on each ring. The total weight of the pearls on each of these four rings is 17, 13, 7 and 5 grams respectively. What is the weight, in grams, of the unused pearl?

A 1 B 2 C 3 D 4 E 5

11. Each region in the figure is to be coloured with one of four colours: red (R), green (G), orange (O) or yellow (Y). The colours of only three regions are shown. Any two regions that touch must have different colours. The colour of the region X is:



A red B orange C green D yellow
E impossible to determine

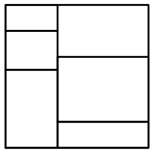
12. A teacher has a list of marks: 17, 13, 5, 10, 14, 9, 12, 16. Which two marks can be removed without changing the mean?

A 12 and 17 B 5 and 17 C 9 and 16 D 10 and 12 E 10 and 14

13. In three home games, Barcelona scored three goals and let in one goal. In these three games, Barcelona won one game, drew one game and lost one game. What was the score in the game Barcelona won?

A 2-0 B 3-0 C 1-0 D 2-1 E 0-1

14. A square piece of paper is cut into six rectangular pieces as shown in the diagram. When the lengths of the perimeters of the six rectangular pieces are added together, the result is 120 cm. What is the area of the square piece of paper?


not to scale

A 48 cm^2 B 64 cm^2 C 110.25 cm^2 D 144 cm^2 E 256 cm^2

15. Lali draws a line segment DE of length 2 cm on a piece of paper. How many different points F can she draw on the paper so that the triangle DEF is right-angled and has an area of 1 cm^2?

A 2 B 4 C 6 D 8 E 10

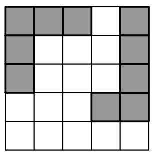
16. The positive number a is less than 1, and the number b is greater then 1. Which of the following numbers has the largest value?

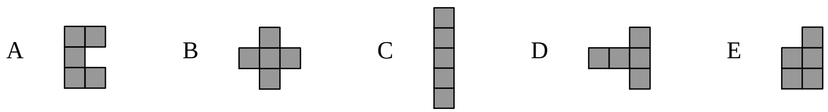
A $a \times b$ B $a + b$ C $a \div b$ D b E The answer depends on a and b.

17. The five-digit number '24X8Y' is divisible by 4, 5 and 9. What is the sum of the digits X and Y?

A 13 B 10 C 9 D 5 E 4

18. Lina has placed two shapes on a 5 × 5 board, as shown in the picture on the right. Which of the following five shapes should she place on the empty part of the board so that none of the remaining four shapes will fit in the empty space that is left? (The shapes may be rotated or turned over, but can only be placed so that they cover complete squares.)





Solutions to the Olympiad Maclaurin Paper

1. How many positive integers leave a remainder of 31 when divided into 2011?

Solution

If an integer leaves a remainder of 31 when divided into 2011, then it divides exactly into 2011 − 31 = 1980. Also, in order to leave a remainder of 31, the integer itself needs to be greater than 31.

Now $1980 = 2^2 \times 3^2 \times 5 \times 11$. Hence any divisor of 1980 may be obtained by choosing one term from each of the following four lists and multiplying them together:

$$1,\ 2,\ 2^2;$$

$$1,\ 3,\ 3^2;$$

$$1,\ 5;$$

$$1,\ 11.$$

There are $3 \times 3 \times 2 \times 2 = 36$ ways of choosing the terms and therefore 1980 has 36 divisors.

The divisors of 1980 which are less than or equal to 31 are

1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 15, 18, 20, 22 and 30,

that is, 15 divisors in all. Hence the number of divisors of 1980 which are greater than 31 is 36 − 15 = 21.

Therefore there are 21 positive integers which leave a remainder of 31 when divided into 2011.

2. I have 44 socks in my drawer, each either red or black. In the dark I randomly pick two socks, and the probability that they do not match is $\frac{192}{473}$.

How many of the 44 socks are red?

Solution

Let there be r red socks and so there are $44 - r$ black socks.

To pick a non-matching pair, I would either have to choose a red followed by a black or vice-versa. The probability of choosing a red and then a black is

$$\frac{r}{44} \times \frac{44 - r}{43}$$

and the probability of choosing a black and then a red is

$$\frac{44 - r}{44} \times \frac{r}{43}.$$

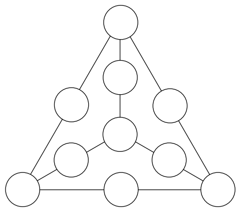
These two probabilities are identical, and hence the probability of picking a non-matching pair is

$$\frac{2r(44 - r)}{44 \times 43} = \frac{192}{473}.$$

This equation simplifies to $r(44 - r) = 384$, that is, $r^2 - 44r + 384 = 0$. After factorising, we obtain $(r - 12)(r - 32) = 0$ and so $r = 12$ or $r = 32$. In other words,

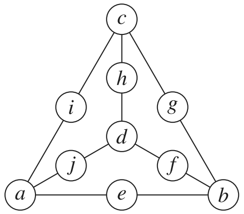
6. Sam wishes to place all the numbers from 1 to 10 in the circles, one to each circle, so that each line of three circles has the same total.

Prove that Sam's task is impossible.



Solution

Let T be the common total and let the numbers in the circles be a to j, as shown in the figure. Note that a, b, c, d are the numbers which occur in three lines.



Finding the sum of the six lines of three numbers, we obtain

$$3(a + b + c + d) + (e + f + g + h + i + j) = 6T. \qquad (6.1)$$

Now the sum of all the numbers from 1 to 10 equals 55, so that

$$(a + b + c + d) + (e + f + g + h + i + j) = 55.$$

Hence equation 6.1 may be rewritten

$$2(a + b + c + d) + 55 = 6T. \qquad (6.2)$$

But 55 is odd and the other two terms in equation 6.2 are even, which is not possible. We deduce that Sam's task is impossible.

19. Three blackbirds, Isaac, Max and Oscar, are each sitting on their own nest. Isaac says: “I'm more than twice as far away from Max as I am from Oscar”. Max says: “I'm more than twice as far away from Oscar as I am from Isaac”. Oscar says: “I'm more than twice as far away from Max as I am from Isaac”. At least two of them are telling the truth. Who is lying?

A Isaac B Max C Oscar D None of them E Impossible to tell

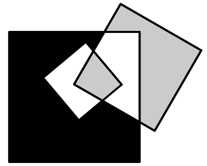
20. Myshko shot at a target. When he hit the target he only scored 5, 8 and 10. Myshko hit 8 and 10 the same number of times. He scored 99 points in total, and 25% of his shots missed the target. How many times did Myshko shoot at the target?

A 10 B 12 C 16 D 20 E 24

21. The diagram on the right shows a square with side 3 cm inside a square with side 7 cm and another square with side 5 cm which intersects the first two squares. What is the difference between the area of the black region and the total area of the grey regions?

A 0 cm^2 B 10 cm^2 C 11 cm^2 D 15 cm^2

E more information needed



not to scale

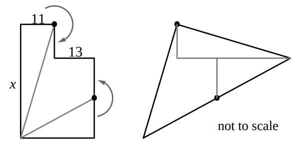
22. In a convex quadrilateral $ABCD$ with $AB = AC$, the following angles are known: $\angle BAD = 80°$, $\angle ABC = 75°$ and $\angle ADC = 65°$. What is the size of $\angle BDC$?

A 10° B 15° C 20° D 30° E 45°

23. In the expression $\dfrac{K \times A \times N \times G \times A \times R \times O \times O}{G \times A \times M \times E}$, the same letter stands for the same non-zero digit and different letters stand for different digits. What is the smallest positive integer value of the expression?

A 1 B 2 C 3 D 5 E 7

24. The first diagram on the right shows a shape constructed from two rectangles. The lengths of two sides are marked: 11 and 13. The shape is cut into three parts and the parts are rearranged, as shown in the second diagram on the right. What is the length marked x?



A 37 B 38 C 39 D 40 E 41

25. Mark plays a computer game on a 4×4 grid. Initially the 16 cells are all white; clicking one of the white cells changes it to either red or blue. Exactly two cells will become blue and they have a side in common. The aim is to make both blue cells appear in as few clicks as possible. What is the largest number of clicks Mark will ever need to make?

A 8 B 9 C 10 D 11 E 12

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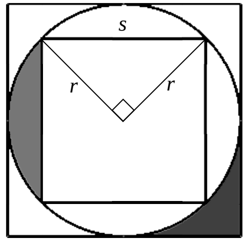


Applying Pythagoras' theorem to this triangle, we obtain $s^2 = r^2 + r^2 = 2r^2$. Thus the total area between the circle and the inner square is

$$\pi r^2 - s^2 = \pi r^2 - 2r^2 = (\pi - 2)r^2.$$

Since the figure has rotational symmetry of order four, the ratio of the darker shaded area to the lighter shaded area is therefore

$$\frac{(4-\pi)r^2}{4} : \frac{(\pi-2)r^2}{4} = (4-\pi) : (\pi - 2).$$

5. In how many distinct ways can a cubical die be numbered from 1 to 6 so that consecutive numbers are on adjacent faces? Numberings that are obtained from each other by rotation or reflection are considered indistinguishable.

Solution

In any rotation or reflection, adjacent faces remain adjacent, and opposite faces remain opposite. We know that face 1 is adjacent to face 2, and can therefore be opposite to 3, 4, 5 or 6. Without loss of generality we may take 1 as the base. Now consider the four possible top faces in turn.

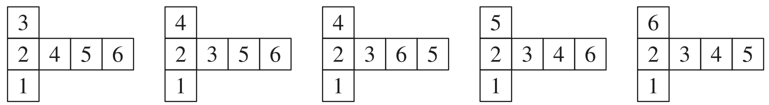
3 at the top Then 2, 4, 5, 6 form the sides. Now 5 has to be adjacent to 4 and 6 is opposite 2, so there is only one such cube possible.

4 at the top Then 2, 3, 5, 6 form the sides. Now 2 and 3 are adjacent, as are 5 and 6, so there are two possible cubes, because we can have 2, 3, 5, 6 (with 2 opposite 5) or 2, 3, 6, 5 (with 2 opposite 6) in order round the cube.

5 at the top Then 2, 3, 4, 6 form the sides. Since 3 is adjacent to 2 and 4, and hence opposite 6, there is just one possibility.

6 at the top Then 2, 3, 4, 5 form the sides. We must have 2 opposite 4, and 3 opposite 5, so there is just one possibility.

Thus there are five distinct ways that a cubical die can be numbered from 1 to 6 so that consecutive numbers are on adjacent faces. The corresponding nets are:



3. A particular four-digit number N is such that:

(a) the sum of N and 74 is a square; and

(b) the difference between N and 15 is also a square.

What is the number N?

Solution

Let

$$N + 74 = x^2 \qquad (3.1)$$

$$\text{and} \qquad N - 15 = y^2, \qquad (3.2)$$

where x and y are different positive integers.

Subtracting equation 3.2 from equation 3.1 gives $89 = x^2 - y^2$. Hence

$$89 = (x - y)(x + y). \qquad (3.3)$$

Now x and y are integers, so equation 3.3 gives a factorisation of 89. But 89 is a prime number, so the only possible factors are 1 and 89. Since $x + y > x - y$ we therefore have

$$x + y = 89$$

$$x - y = 1.$$

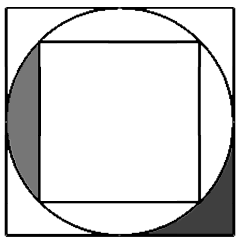
Adding these equations gives $2x = 90$, thus $x = 45$.

Substituting for x in equation 3.1, we obtain $N = 45^2 - 74 = 2025 - 74 = 1951$.

Check: we may also find $y = 44$ and substitute in equation 3.2, to obtain $N = 44^2 + 15 = 1936 + 15 = 1951$.

4. A square just fits within a circle, which itself just fits within another square, as shown in the diagram.

Find the ratio of the two shaded areas.



Solution

Let r be the radius of the circle, so that the outer square has side-length $2r$. Thus the total area between the circle and the outer square is

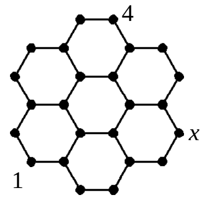
$$(2r)^2 - \pi r^2 = (4 - \pi)r^2.$$

Let the side of the inner square be s. Two radii and one side of this square form a right-angled triangle, as shown.

1. Given that $P = 2 \times 3 + 3 \times 4 + 4 \times 5$, $Q = 2^2 + 3^2 + 4^2$ and $R = 1 \times 2 + 2 \times 3 + 3 \times 4$, which of the following statements is true?

A $Q < P < R$ B $P < Q = R$ C $P < Q < R$ D $R < Q < P$ E $Q = P < R$

2. The figure shows a hexagonal lattice. Numbers are to be placed at each of the dots • in such a way that the sum of the two numbers at the ends of each segment is always the same. Two of the numbers are already given. What number is x?



A 1 B 2 C 3 D 4 E 5

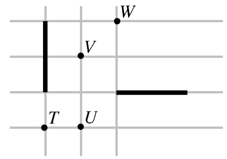
3. A rectangular mosaic with area 360 cm^2 is made from square tiles, all of which are the same size. The mosaic is 24 cm high and 5 tiles wide. What is the area of each tile in cm^2 ?

A 1 B 4 C 9 D 16 E 25

4. Tomas writes down all 4-digit numbers whose digits add up to four. If he writes these numbers in descending order, which position will the number 2011 occupy?

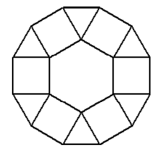
A 6th B 7th C 8th D 9th E 10th

5. One of the line segments shown on the grid is the image produced by a rotation of the other line segment. Which of the points T, U, V, W could be the centre of such a rotation?



A only T B only U C either of U and W
D any of U, V and W E any of T, U, V and W

6. The diagram shows a shape made from a regular hexagon of side one unit, six triangles and six squares. What is the perimeter of the shape?



A $6(1 + \sqrt{2})$ B $6(1 + \frac{1}{2}\sqrt{3})$ C 12 D $6 + 3\sqrt{2}$ E 9

7. Three normal dice are placed one on top of another, with the bottom die standing on a table. Where the two lower dice meet, the spots on the two touching faces add to five; similarly where the two higher dice meet, the spots on the two touching faces add to five. One of the visible faces on the bottom die shows just one spot. How many spots are on the top face of the top die?

A 2 B 3 C 4 D 5 E 6

8. In a certain month last year, there were five Mondays, five Tuesdays and five Wednesdays. In the month before there had been exactly four Sundays. Which of the following were included in the month after?

A exactly four Fridays B exactly four Saturdays C exactly five Wednesdays
D exactly five Saturdays E exactly five Sundays

9. Michael, Fernando and Sebastian had a race. Immediately after the start Michael was in the lead with Fernando second and Sebastian last. During the race Michael overtook, or was overtaken by Fernando a total of 9 times; similarly Fernando and Sebastian interchanged places 10 times, and Michael and Sebastian interchanged places 11 times. In what order, first to last, did they finish?

A Michael, Fernando, Sebastian B Fernando, Sebastian, Michael
C Sebastian, Michael, Fernando D Sebastian, Fernando, Michael
E Fernando, Michael, Sebastian

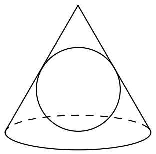
10. Given that $9^n + 9^n + 9^n = 3^{2011}$, what is the value of n?

A 1005 B 1006 C 2010 D 2011 E 6033

11. Ulf has two cubes, with sides of length a cm and $a + 1$ cm. The larger cube is full of water and the smaller cube is empty. Ulf now fills the smaller cube with water from the larger cube, leaving 217 ml in the larger cube. How much water is then in the smaller cube, in ml?

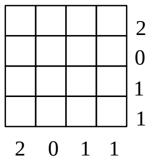
A 125 B 243 C 512 D 729 E 1331

12. A marble with radius 15 cm fits exactly under a cone as shown in the diagram. The slant height of the cone is equal to the diameter of its base. What is the height of the cone in cm?



A 45 B $25\sqrt{3}$ C $30\sqrt{2}$ D 60 E $60(\sqrt{3} - 1)$

13. Barbara wants to place draughts on a 4×4 board in such a way that the number of draughts in each row is equal to the number shown at the end of the row, and the number of draughts in each column is equal to the number shown at the bottom of the column. No more than one draught is to be placed in any cell. In how many ways can this be done?



A 1 B 2 C 3 D 4 E 5

14. How many numbers appear in the longest run of consecutive 3-digit numbers each of which has at least one odd digit?

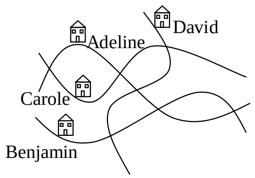
A 1 B 10 C 100 D 110 E 111

15. Nik wants to write integers in the cells of a 3×3 table so that the sum of the numbers in any 2×2 square is 10. He has already written five numbers in the table as shown. What is the sum of the four missing numbers?

1		0
	2	
4		3

A 9 B 10 C 11 D 12 E 13

16. During a rough sailing trip, Jacques tried to sketch a map of his village. He managed to draw the four streets, the seven places where they cross and the houses of his friends. The houses are marked on the correct streets, and the intersections are correct, however, in reality, Arrow Street, Nail Street and Ruler Street are all absolutely straight. The fourth street is Curvy Street. Who lives on Curvy Street?



A Adeline B Benjamin C Carole
D David E It is impossible to tell without a better map

Solutions to the Olympiad Hamilton Paper

1 If Julie gave £12 to her brother Garron then he would have half the amount that she would have. If instead Garron gave £12 to his sister Julie then she would have three times the amount that he would have.

How much money do they each have?

Solution

Let Julie and Garron have £J and £G respectively. From the information in the question we may form the equations

$$J - 12 = 2(G + 12) \tag{1.1}$$

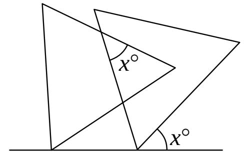
$$J + 12 = 3(G - 12). \tag{1.2}$$

Subtracting equation 1.1 from equation 1.2 gives $24 = G - 60$; thus $G = 84$.

Substituting into equation 1.1 then gives $J - 12 = 2(84 + 12) = 192$. Therefore $J = 204$.

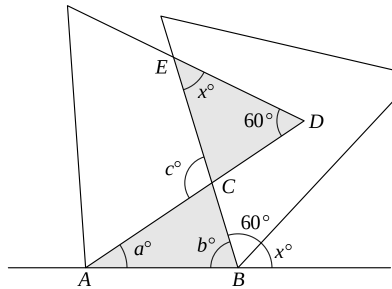
Hence Julie has £204 and Garron has £84.

2. The diagram shows two equilateral triangles. The angles marked $x°$ are equal.
Prove that $x > 30$.



Solution

Each of the angles in an equilateral triangle is equal to 60°. We add two 60° angles to the figure, label three other angles $a°$, $b°$ and $c°$, and label some points, as shown.



At B, since angles on the straight line add up to 180° we have $b + 60 + x = 180$, that is,

$$b = 120 - x. \tag{2.1}$$

Now the angle labelled $c°$ is an exterior angle of triangle CDE, so that $c = x + 60$; it is also an exterior angle of triangle ABC, so that $c = a + b$. Hence $a + b = x + 60$ and therefore, using equation 2.1, we have $a = x + 60 - (120 - x) = 2x - 60$.

But $a > 0$ for the given configuration to occur, hence $2x - 60 > 0$, that is, $x > 30$, as required.

Notice that any move by the bug increases the row number by at most 1. So in order to get from row 1 to row 14, the bug must visit all fourteen rows on the way, so must visit at least fourteen triangles. However, the bug can do it in fourteen, for example, by following the path depicted in figure 2. This settles part (a).

For part (b), note that any path by the bug that visits only fourteen triangles must increase the row number with every step. However, from any odd-numbered row (on an upwards-pointing triangle) the only way to increase the row number is to move directly downwards.

That means we might as well join each upwards-pointing triangle with the triangle below it to form a diamond-shaped cell (as shown in figure 3): they come as a pair.

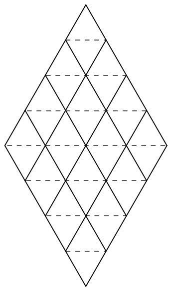


Figure 3

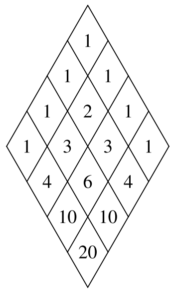


Figure 4

Using this, we can count the number of different ways of reaching each cell. Clearly there is only one way to reach the top cell.

For every cell lower down, one must first reach one of the neighbouring cells above. So the number of different ways of reaching each cell is the number of ways of reaching it from above and to the left, plus the number of ways of reaching it from above and to the right. Continuing in this way, we get the table of numbers seen in figure 4.

Hence the answer to part (b), the number of ways of reaching the bottom cell, is 20.

Alternative method for (a)

An alternative method of finding the minimum number of triangles visited is to consider the straight lines across the diagram. There are seven horizontal lines, and three lines in each diagonal direction. This makes a total of 13 straight lines. The bug wishes to cross all of them, so must cross at least 13 edges. In crossing 13 edges, the bug must visit at least 14 triangles.

17. The numbers x and y are both greater than 1. Which of the following fractions has the greatest value?

A $\dfrac{x}{y+1}$ B $\dfrac{x}{y-1}$ C $\dfrac{2x}{2y+1}$ D $\dfrac{2x}{2y-1}$ E $\dfrac{3x}{3y+1}$

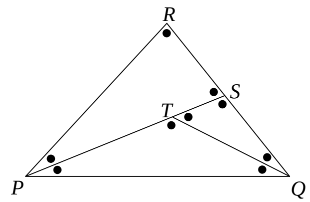
18. Simone has a cube with sides of length 10 cm, and a pack of identical square stickers. She places one sticker in the centre of each face of the cube, and one across each edge so that the stickers meet at their corners, as shown in the diagram. What is the total area in cm^2 of the stickers used by Simone?

A 150 B 180 C 200 D 225 E 300

19. Rafael writes down a 5-digit number whose digits are all distinct, and whose first digit is equal to the sum of the other four digits. How many 5-digit numbers with this property are there?

A 72 B 144 C 168 D 216 E 288

20. In triangle PQR, a point S is chosen on the line segment QR, then a point T is chosen on the line segment PS. Considering the nine marked angles, what is the smallest number of different values that these nine angles could take?



A 2 B 3 C 4 D 5 E 6

21. Xerxes chooses a positive integer x, and Yasmin chooses a positive integer y, such that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3}$. In how many ways could they choose these numbers?

A 1 B 2 C 3 D 4 E 5

22. C_1 is a circle of radius r. PQ is a chord of this circle. C_2 is a circle with diameter PQ and which passes through the centre of C_1. What is the area of the part of the circle C_2 which is outside the circle C_1?

A $\dfrac{1}{2}r^2$ B $\dfrac{\sqrt{3}\pi}{12}r^2$ C $\dfrac{\pi}{6}r^2$ D $\dfrac{\sqrt{3}}{4}r^2$ E $\dfrac{1}{\sqrt{2}}r^2$

23. Hassan selects four edges of a cube in such a way that none of the edges share a common vertex. How many different ways are there for Hassan to do this?

A 6 B 8 C 9 D 12 E 18

24. Barbara has a new challenge. She places draughts on a 5×5 board in such a way that each 3×3 square contains exactly n draughts. No more than one draught is placed in any cell. Given that $0 < n < 9$, what are the possible values of n?

A 1 B 1 and 8 C 1, 2, 7 and 8 D 1, 2, 3, 6, 7 and 8
E All whole numbers 1 to 8 inclusive

25. This morning, the two turtles Tor and Tur multiplied their ages together, correctly obtaining 1188. When they multiply their ages together on this day next year, which of the following will definitely not be a factor of the product?

A 19 B 21 C 23 D 25 E 27

The United Kingdom Mathematics Trust

Intermediate Mathematical Olympiad and Kangaroo (IMOK)

Olympiad Cayley/Hamilton/Maclaurin Papers

Thursday 17th March 2011

READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, protractors and squared paper is forbidden.** Rulers and compasses may be used.
3. Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Cover Sheet on top.
4. Start each question on a fresh A4 sheet.
   You may wish to work in rough first, then set out your final solution with clear explanations and proofs. ***Do not hand in rough work.***
5. Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as π, fractions, or square roots, if appropriate, but NOT decimal approximations.
6. Give full written solutions, including mathematical reasons as to why your method is correct.
   Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.
7. **These problems are meant to be challenging!** The earlier questions tend to be easier; the last two questions are the most demanding.
   Do not hurry, but spend time working carefully on one question before attempting another. Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

The United Kingdom Mathematics Trust is a Registered Charity.
Enquiries should be sent to: Maths Challenges Office,
School of Mathematics Satellite, University of Leeds, Leeds, LS2 9JT.
(Tel. 0113 343 2339)
http://www.ukmt.org.uk

5 Solve the equation $5a - ab = 9b^2$, where a and b are positive integers.

Solution

Notice that the right-hand side $9b^2$ is always positive, since the square b^2 is always positive. However, the left-hand side $5a - ab = a(5 - b)$ is only positive for $b \leqslant 4$. So, given that b is a positive integer, we can consider four cases separately, namely $b = 1, 2, 3, 4$.

If $b = 1$, then the equation becomes $(5 - 1)a = 9 \times 1^2$, that is, $4a = 9$. This has no solution for a positive integer a.

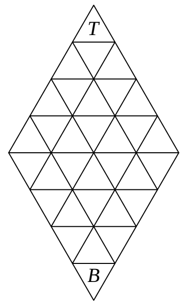
If $b = 2$, then the equation becomes $(5 - 2)a = 9 \times 2^2$, that is, $3a = 36$, so that $a = 12$. This gives the solution $a = 12$, $b = 2$.

If $b = 3$, then the equation becomes $(5 - 3)a = 9 \times 3^2$, that is, $2a = 81$. This has no solution for a positive integer a.

If $b = 4$, then the equation becomes $(5 - 4)a = 9 \times 4^2$, that is, $a = 144$. This gives the solution $a = 144$, $b = 4$.

Thus the solutions are $a = 12$, $b = 2$ and $a = 144$, $b = 4$.

6 A bug starts in the small triangle T at the top of the diagram. She is allowed to eat through a neighbouring edge to get to a neighbouring small triangle. So at first there is only one possible move (downwards), and only one way to reach this new triangle.



(a) How many triangles, including T and B, must the bug visit if she is to reach the small triangle B at the bottom using a route that is as short as possible?

(b) How many different ways are there for the bug to reach B from T by a route of this shortest possible length?

Solution

Shade the downwards-pointing triangles grey and leave the upwards-pointing triangles white. We separate the triangles into rows, which alternate between white and grey triangles. So, as shown in figure 1, row 1 consists of just the top triangle T, row 2 consists of the grey triangle below T, row 3 consists of the two white triangles next to it, and so on. Finally, row 14 consists of the bottom triangle B.

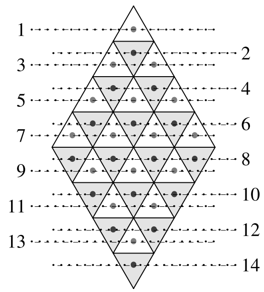


Figure 1

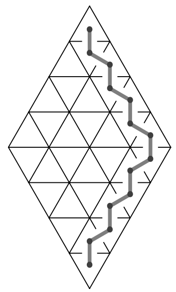


Figure 2

3 At dinner on a camping expedition, each tin of soup was shared between 2 campers, each tin of meatballs was shared between 3 campers and each tin of chocolate pudding was shared between 4 campers. Each camper had all three courses and all tins were emptied. The camp leader opened 156 tins in total.

How many campers were on the expedition?

Solution

Each camper eats half a tin of soup, one third of a tin of meatballs, and a quarter of a tin of chocolate pudding. This is a total of

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

tins of food per person.

If there are N people, they thus use $\frac{13}{12}N$ tins of food in total. This gives us the equation

$$\frac{13}{12}N = 156,$$

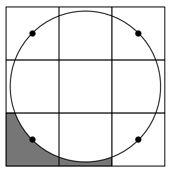
so that

$$N = \frac{12}{13} \times 156$$
$$= 144.$$

Therefore there are 144 campers on the expedition.

4 The diagram shows nine 1 cm × 1 cm squares and a circle. The circle passes through the centres of the four corner squares.

What is the area of the shaded region—inside two squares but outside the circle?



Solution

Consider the region of the big square which lies outside the circle. The lines in the figure divide the region into eight parts, four in the four corner squares, which are all identical by symmetry, and four in the four edge-centre squares, which are again all identical by symmetry.

The shaded area contains one part of each sort, and so takes up exactly a quarter of the difference between the big square and the circle, which we can work out.

The big square has area 9 cm^2, being made up of 9 squares each measuring $1\text{cm} \times 1\text{cm}$.

By Pythagoras' theorem, the diagonal of a $1\text{cm} \times 1\text{cm}$ square has length $\sqrt{2}$ cm. The radius of the circle is the distance from the centre of the middle square to the centre of a corner square, so is the length of one diagonal in total.

Hence the area of the circle is $\pi\left(\sqrt{2}\right)^2 = 2\pi$.

Putting this all together, we see that the entire region outside the circle has area $9 - 2\pi$, and so the shaded region has area

$$\frac{9 - 2\pi}{4}.$$

- *Do not hurry, but spend time working carefully on one question before attempting another.*
- *Try to finish whole questions even if you cannot do many.*
- *You will have done well if you hand in full solutions to two or more questions.*

- *Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as π, fractions, or square roots, if appropriate, but NOT decimal approximations*

- *Give full written solutions, including mathematical reasons as to why your method is correct.*
- *Just stating an answer, even a correct one, will earn you very few marks.*
- *Incomplete or poorly presented solutions will not receive full marks.*

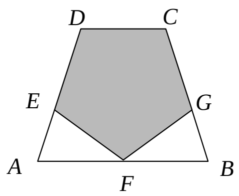
- ***Do not hand in rough work.***

Olympiad Cayley Paper

All candidates must be in *School Year 9 or below* (England and Wales), *S2 or below* (Scotland), or *School Year 10 or below* (Northern Ireland).

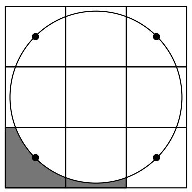
1. A palindromic number is one which reads the same when its digits are reversed, for example 23832.
   What is the largest six-digit palindromic number which is exactly divisible by 15?

2. The diagram shows a regular pentagon *CDEFG* inside a trapezium *ABCD*.
   Prove that $AB = 2 \times CD$.



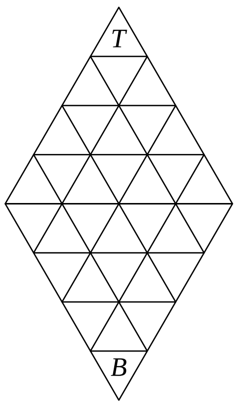
3. At dinner on a camping expedition, each tin of soup was shared between 2 campers, each tin of meatballs was shared between 3 campers and each tin of chocolate pudding was shared between 4 campers. Each camper had all three courses and all tins were emptied. The camp leader opened 156 tins in total.
   How many campers were on the expedition?

4. The diagram shows nine 1 cm × 1 cm squares and a circle. The circle passes through the centres of the four corner squares.
   What is the area of the shaded region—inside two squares but outside the circle?



5. Solve the equation $5a - ab = 9b^2$, where a and b are positive integers.

6. A bug starts in the small triangle *T* at the top of the diagram. She is allowed to eat through a neighbouring edge to get to a neighbouring small triangle. So at first there is only one possible move (downwards), and only one way to reach this new triangle.
   (a) How many triangles, including *T* and *B*, must the bug visit if she is to reach the small triangle *B* at the bottom using a route that is as short as possible?
   (b) How many different ways are there for the bug to reach *B* from *T* by a route of this shortest possible length?



Solutions to the Olympiad Cayley Paper

1 A palindromic number is one which reads the same when its digits are reversed, for example 23832.

What is the largest six-digit palindromic number which is exactly divisible by 15?

Solution

We note that being divisible by 15 is the same as being divisible by 3 and by 5.

We also note that a number is divisible by 5 if, and only if, the units digit is 0 or 5.

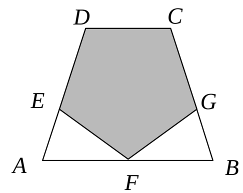
However, our number cannot end in a 0. Indeed, in mathematics every number begins with a non-zero digit. A palindrome has equal first and last digits, so the last digit is non-zero.

Hence we seek a particular six-digit palindrome which begins and ends in 5, and which is divisible by 3.

The largest six-digit palindromes beginning and ending in 5 have the form 59*dd*95, for some digit d. This is divisible by 3 when the digit-sum is a multiple of 3 and therefore $5 + d$ is divisible by 3. So $d = 1, 4$, or 7. Hence the number required is 597795.

2 The diagram shows a regular pentagon *CDEFG* inside a trapezium *ABCD*.

Prove that $AB = 2 \times CD$.



Solution

First of all, we amass some basic facts.

Since the exterior angles of any polygon add up to 360°, the exterior angles of a regular pentagon are each $360° \div 5 = 72°$. Thus the interior angles are each $180° - 72° = 108°$.

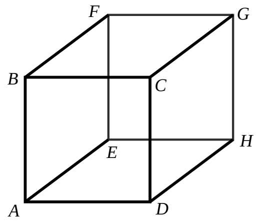
Now, we claim that triangle *AEF* is isosceles. Indeed, $\angle AEF = 72°$ since it is an exterior angle of the pentagon. Also, since *AB* and *DC* are parallel, $\angle EAF$ is supplementary to $\angle EDC$ (they are allied angles). So $\angle EAF = 180° - 108° = 72°$ too.

This gives us that $AF = EF$.

Similarly, triangle *BGF* is isosceles, and $BF = GF$.

But now $AB = AF + FB = EF + FG = CD + CD$ because the sides of a regular pentagon are equal, so $AB = 2 \times CD$ as required.

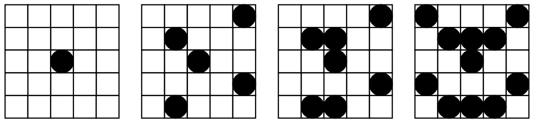
23. C The four edges use distinct vertices, so all eight vertices will be used for each set. If we consider vertex *A*, there are three choices of edge: *AB*, *AD*, *AE*.



Start with *AB*. This leaves two choices from *D*: *DC* or *DH*. Choosing *DC* means we have two choices for the final pairs (*GH* and *EF*, or *EH* and *FG*). Choosing *DH* means we must choose *EF* and *GC*.

Hence there are 3 sets of edges when we start with *AB*. There will also be 3 sets if we start with *AD* or *AE*, and these 9 sets will all be distinct, so the answer is 9.

24. E It is worth noting that if it is possible to place draughts and get n of them in each 3×3 square, then there are $9 - n$ spaces in each 3×3 square. Thus by swapping draughts for spaces and spaces for draughts, Barbara could also get $9 - n$ draughts in each 3×3 square. So it is sufficient to show that it is possible to achieve 1, 2, 3 and 4 draughts, as demonstrated in the diagrams below.



25. E The prime factorisation of 1188 is $2^2 \times 3^3 \times 11$, and the current ages of Tor and Tur could feasibly be any combination of these factors. Assuming Tor is younger than Tur, their current ages could be:

Tor:	1,	2,	3,	4,	6,	9,	11,	12,	18,	22,	27,	33
Tur:	1188,	594,	396,	297,	198,	132,	108,	99,	66,	54,	44,	36

Their ages next year will be one more than their current age, so could be:

Tor:	2,	3,	4,	5,	7,	10,	12,	13,	19,	23,	28,	34
Tur:	1189,	595,	397,	298,	199,	133,	109,	100,	67,	55,	45,	37

We are looking for factors of the products of these possible pairs of ages. Now 19 and 23 are in this list, so they might be a factor. Also 21 divides 28×45 and 25 divides 100; so each of them might be a factor. Could 27 be a factor? Well the only numbers listed above which are multiples of 3 are 3, 12 and 45. None of these is paired with a multiple of 3. So the highest possible power of 3 in the product would be in 28×45 – and that product is divisible by 9 but not by 27. Hence 27 is not a possible factor of the product of their ages.

Olympiad Hamilton Paper

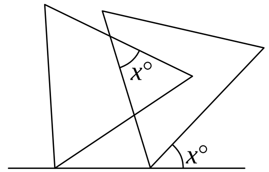
All candidates must be in *School Year 10* (England and Wales), *S3* (Scotland), or *School Year 11* (Northern Ireland).

1. If Julie gave £12 to her brother Garron, then he would have half the amount that she would have. If instead Garron gave £12 to his sister Julie, then she would have three times the amount that he would have.

   How much money do they each have?

2. The diagram shows two equilateral triangles. The angles marked $x°$ are equal.

   Prove that $x > 30$.



3. A particular four-digit number N is such that:

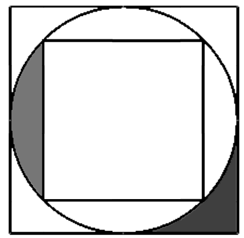
   (a) the sum of N and 74 is a square; and

   (b) the difference between N and 15 is also a square.

   What is the number N?

4. A square just fits within a circle, which itself just fits within another square, as shown in the diagram.

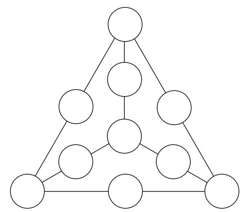
   Find the ratio of the two shaded areas.



5. In how many distinct ways can a cubical die be numbered from 1 to 6 so that consecutive numbers are on adjacent faces? Numberings that are obtained from each other by rotation or reflection are considered indistinguishable.

6. Sam wishes to place all the numbers from 1 to 10 in the circles, one to each circle, so that each line of three circles has the same total.

   Prove that Sam's task is impossible.



Olympiad Maclaurin Paper

All candidates must be in *School Year 11* (England and Wales), *S4* (Scotland), or *School Year 12* (Northern Ireland).

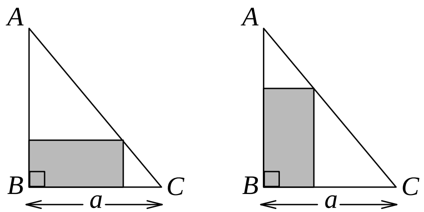
1. How many positive integers leave a remainder of 31 when divided into 2011?

2. I have 44 socks in my drawer, each either red or black. In the dark I randomly pick two socks, and the probability that they do not match is $\frac{192}{473}$.

   How many of the 44 socks are red?

3. The diagrams show a rectangle that just fits inside right-angled triangle ABC in two different ways. One side of the triangle has length a.

   Prove that the perimeter of the rectangle is $2a$.

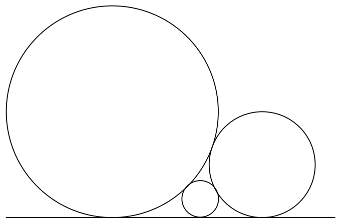


4. How many solutions are there to the equation $x^2 + y^2 = x^3$, where x and y are positive integers and x is less than 2011?

5. Three circles touch the same straight line and touch each other, as shown.

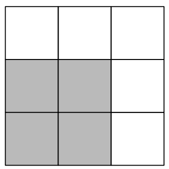
   Prove that the radii a, b and c, where c is smallest, satisfy the equation

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}.$$



6. The numbers 1 to 9 are placed in the cells of a 3×3 square grid, one to each cell. In each of the four 2×2 blocks of adjacent cells, such as the one shaded, the four numbers have the same total T.

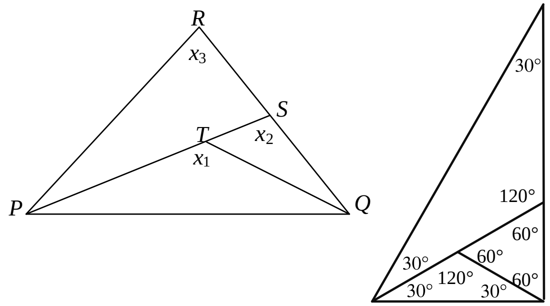
   What is the maximum possible value of T?



19. C The first digit is equal to the sum of the other four digits, so the sum of the last four digits must be less than 10. The seven sets of four distinct digits whose total is less than 10 are: {0,1,2,3}, {0,1,2,4}, {0,1,2,5}, {0,1,2,6}, {0,1,3,4}, {0,1,3,5}, {0,2,3,4}. Once we have picked four digits, they can be arranged in 24 ways (4 choices for the first, 3 choices for the second, 2 for the third and 1 for the last gives $4 \times 3 \times 2 \times 1 = 24$ arrangements).

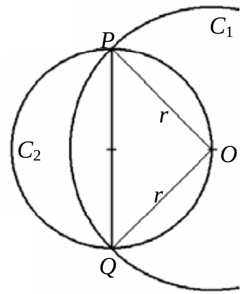
So there are $7 \times 24 = 168$ possible numbers.

20. B Label angles x_1, x_2, x_3, as shown in the first diagram. Since an exterior angle of a triangle equals the sum of the interior opposite angles, x_1 is greater than x_2 which in turn is greater than x_3. So we must have at least three different values for the nine angles. The second diagram shows a triangle where we obtain precisely three different values.



21. C If x or y is less than 4, then $\frac{1}{x} + \frac{1}{y} > \frac{1}{3}$ and if x and y are both greater than 6, then $\frac{1}{x} + \frac{1}{y} < \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. So we need at least one of x, y to be 4, 5, or 6. The other fraction will be equal to $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, or $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, or $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$. Since $\frac{2}{15}$ cannot be expressed as a unit fraction, the possibilities are $x = 4, y = 12$; $x = 12, y = 4$; $x = 6, y = 6$.

22. A Let A_1 be the area of the small circle C_2; let A_2 be the area of the sector OPQ of the circle C_1 and let A_3 be the area of the triangle OPQ. Then the desired area is $\frac{1}{2}A_1 - (A_2 - A_3)$. Angle POQ is 90° (angle in a semicircle) so by Pythagoras, $PQ^2 = r^2 + r^2$, giving $PQ = \sqrt{2}r$, and so the radius of the small circle is $\frac{1}{2}\sqrt{2}r$.



Then $A_1 = \pi\left(\frac{\sqrt{2}r}{2}\right)^2 = \pi\left(\frac{2r^2}{4}\right) = \frac{\pi r^2}{2}$,

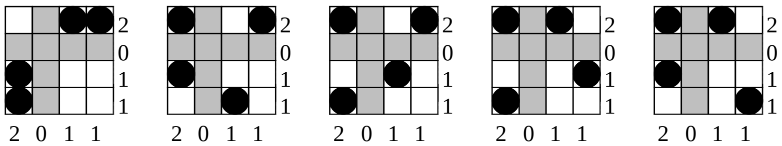
$A_2 = \frac{\pi r^2}{4}$ and $A_3 = \frac{1}{2} \times r \times r = \frac{r^2}{2}$.

So the desired area is $\frac{1}{2} \times \frac{\pi r^2}{2} - \frac{\pi r^2}{4} + \frac{r^2}{2} = \frac{r^2}{2}$.

13. E Since one row and one column have no draughts, we need only consider the other rows and columns. If there is more than 1 draught in the bottom right 2×2 square, that would mean there were 2 there, one in each of the two columns and rows involved. So there would be no draughts at the top of those two columns and none at the left hand end of those two rows. So the only place where there could be another draught is the top left hand corner. But the top row needs two draughts. Hence there is at most one draught in the bottom right 2×2 square.

If the bottom right 2×2 square has no draughts, then there must be one at the top of each column numbered 1, and one at the start of each row numbered 1.

Otherwise there are four possible places for one draught in the bottom right square, each of which forces the positions of the remaining draughts. Hence there are five possibilities shown below.



14. E The 111 numbers between 289 and 399 inclusive all contain at least one odd digit (288 and 400 do not have odd digits). This occurs again every 200 numbers (from 489, from 689 and from 889). So there are 4 runs of 111 numbers with at least one odd digit, and the gaps between are not big enough to contain a longer run (the gap from 100 to 289 is long enough but contains numbers with no odd digits, e.g. 200).

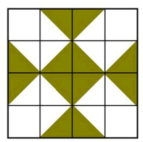
15. D Let the missing numbers be a, b, c and d as shown. The top left 2×2 square adds to 10 so $a + b = 7$. Similarly the bottom right 2×2 square adds to 10 so $c + d = 5$. Hence $a + b + c + d = 12$.

1	a	0
b	2	c
4	d	3

16. A A pair of straight lines intersects at most once, but Adeline's and Carole's roads intersect twice so one of them must be Curvy Street; similarly Adeline's and Benjamin's roads intersect twice so one of them must also be Curvy Street. Therefore Adeline lives on Curvy Street.

17. B We can convert the five fractions into equivalent fractions with the same numerator by multiplying both the numerator and the denominator of the first two by 6, the next two by 3 and the last by 2, giving: $\frac{6x}{6y+6}, \frac{6x}{6y-6}, \frac{6x}{6y+3}, \frac{6x}{6y-3}, \frac{6x}{6y+2}$.
Since x and y are greater than 1, these are all positive, so the fraction with the smallest denominator will have the greatest value. Clearly $6y - 6$ is the smallest, so $\frac{x}{y-1}$ has the greatest value.

18. D By dividing the front face of the cube into 16 congruent squares, it is easily seen that the area of the stickers is $\frac{6}{16}$ of the area of the whole front. There are six faces, each with area 100 cm^2 so the total area of the stickers is $\frac{6}{16} \times 100 \times 6 = 225\ \text{cm}^2$.



Professor Colin Maclaurin

February 1698 − June 1746

The most significant Scottish mathematician and physicist of the eighteenth century, Colin Maclaurin was only 11 years old when he first attended the University of Glasgow. There he came into contact with the Professor of Mathematics, Robert Simson, whose enthusiasm and interest in geometry was to influence the young boy. After graduating in 1713, Maclaurin remained in Glasgow for a further year to read divinity (at that time intending to enter the Presbyterian Church). He then continued to study mathematics and divinity whilst staying with his uncle, the minister at Kilfinnan on Loch Fyne.

Maclaurin was appointed professor of mathematics at Marischal College in the University of Aberdeen in 1717, aged 19. At this time his main interest was in the mathematical and physical ideas of Sir Isaac Newton; he met Newton during a visit to London in 1719, the same year that he was elected a fellow of the Royal Society. Maclaurin also did notable work in geometry, particularly higher plane curves, and his first published work was *Geometria organica, sive descriptio linearum curvarum universalis*, published in 1720. One curve still bears his name, the *Trisectrix of Maclaurin*.

After two years spent travelling in Europe, during which he was awarded a Grand Prize by the Académie des Sciences in Paris for his work on the impact of bodies, Maclaurin took up the post of Professor of Mathematics at the University of Edinburgh in 1725, and remained there for the rest of his career. In 1740 he again received a prize from the Académie des Sciences, this time for a study of the tides. The prize was jointly awarded to four people, including two other famous mathematicians, Leonhard Euler and Daniel Bernoulli.

In 1742, Maclaurin published the *Treatise of fluxions*, in which he uses the special case of Taylor's series now named after him and for which he is best remembered today:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

He also wrote a book of problems for students, *Ane Introduction to the Mathematicks*, one of which was used as the basis for question 2 of the 2004 Hamilton Olympiad Paper:

> Maritus, uxor, et filius habent annos 96, ita ut anni Mariti et filii, simul faciant annos uxoris + 15. Sed uxoris cum filii faciant mariti + 2.

Maclaurin defended mathematical education at universities because of its practical applications, and his own work included gravitation, astronomy, cartography, the structure of honeycombs and the measurement of the volumes of barrels. The field of actuarial science dates back to the calculations he helped to supply when one of the first pension funds was founded in 1743, the Scottish Ministers' Widows' Fund.

Solutions to the European Kangaroo Grey Paper

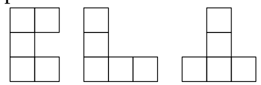
1. **A** The calculation becomes $12 \div 3 - 4 \div 2 = 4 - 2 = 2$.

2. **B** So that there are 8 white stripes, there must be 7 black stripes so that the crossing starts and ends with a white stripe. This makes 15 stripes in all and the total width of the crossing is $15 \times 0.5 = 7.5$ m.

3. **C** The next time that uses the digits 0, 1, 1, 2 in some order is 21:01. This is 50 minutes later.

4. **E** Since the last house on the ‘even’ side is numbered 12, there are 6 houses on the even side. There are therefore 11 houses on the ‘odd’ side and the eleventh odd number is 21.

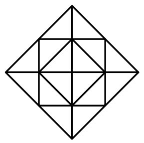
5. **C** The diagram is constructed from four small squares, each of which has at least one side in common with another small square. So Ria must place the extra small square so that it has a side in common with one of the existing squares. Ria can form three new shapes with a line of symmetry, as shown.



6. **A** If Felix caught 6 or more fish on day 3 then, since he caught 12 in total, he must have caught 6 or fewer on the previous two days; but this contradicts what we are told. So he must have caught 5 or fewer on day 3. If he caught 4 or fewer on day 3, then, since the numbers increase day by day, he would have caught fewer than 12 fish in all. So 5 is the only possibility for day 3, with 3 and 4 being the numbers of fish caught on days 1 and 2 respectively.

7. **B** The largest three-digit number whose digits sum to 8 is 800 and the smallest is 107. The sum of these is 907.

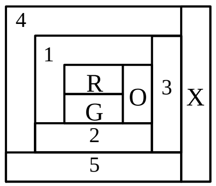
8. **D** The diagram on the right shows how the shape can be dissected into sixteen congruent triangles. The small square has been dissected into four triangles, each of area $6 \div 4 = 1.5\ \text{cm}^2$. The difference in area between the medium and the large square is eight of these triangles, that is $8 \times 1.5 = 12\ \text{cm}^2$.



9. **C** $\dfrac{2011 \times 2.011}{201.1 \times 20.11} \times \dfrac{1000}{1000} = \dfrac{2011 \times (2.011 \times 1000)}{(201.1 \times 10) \times (20.11 \times 100)} = \dfrac{2011 \times 2011}{2011 \times 2011} = 1.$

10. **C** Marie's nine pearls have a total weight of 45 grams. The total weight of pearls on the four rings is 42 grams. Hence the weight of the remaining pearl is 3 grams.

11. **A** Label the regions 1 to 5 as shown in the diagram. Region 1 must be coloured yellow as it touches a red, a green and an orange region. Then region 2 must be coloured red as it touches an orange, a yellow and a green region. Now region 3 must be coloured green as it touches an orange, a yellow and a red region. Then region 4 must be coloured orange as it touches a yellow, a red and a green region. Now region 5 must be coloured yellow as it touches a green, an orange and a red region. Finally, region X must be coloured red as it touches a green, an orange and a yellow region.



7. **E** The touching faces on the two lower dice add to five, so are certainly both less than 5. And since the 1-spot is visible on the lowest die, its upper face could be 2, 3, or 4. We can then proceed, as shown in the table, using the facts that the touching faces add to 5, and opposite faces on a die add to 7. The only possibility for the top face of the top die is 6.

Upper face of bottom die.	Lower face of middle die	Upper face of middle die	Lower face of top die	Top face of top die
2	3	4	1	6
3	2	5	Can’t add to 5	
4	1	6	Can’t add to 5	

Add to 5   Add to 7   Add to 5   Add to 7

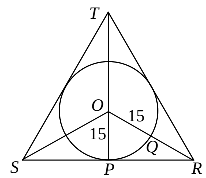
8. **B** The month included four complete weeks, and three more days: Monday, Tuesday, Wednesday, totalling 31 days – the longest possible for any month. Hence it must have begun on a Monday, and ended on the fifth Wednesday. Then the previous month ended on a Sunday, but only had four Sundays, so was at most four weeks long; it must have been February since all other months are more than 28 days long. Then the month after is April, beginning on Thursday. Having 30 days, it will contain four complete weeks, and an extra Thursday and Friday. From the options available, B is the only correct one.

9. **B** One overtaking procedure would swap the positions of two participants, while two would return them to their original positions. Michael and Fernando have an odd number of overtakings, so Fernando ends ahead of Michael; Fernando and Sebastian have an even number of overtakings so Fernando remains ahead; Michael and Sebastian have an odd number of overtakings so Sebastian ends ahead of Michael. They must finish in the order: Fernando, Sebastian, Michael.

10. **A** $9^n = (3^2)^n = 3^{2n}$ so $9^n + 9^n + 9^n = 3 \times 3^{2n} = 3^{2n+1}$, and we must have $2n + 1 = 2011$ so $n = 1005$.

11. **C** The difference between the volume of the two cubes is $(a+1)^3 - a^3 = 3a^2 + 3a + 1 = 217$.
Therefore $3a^2 + 3a - 216 = 0$, and so $a^2 + a - 72 = 0$.
So $(a + 9)(a - 8) = 0$, giving $a = 8$ since a cannot be negative.
Therefore the smaller cube has volume $8^3 = 512\ \text{cm}^3$.

12. **A** Because the slant height of the cone is the same as the diameter of its base, the cross-section of the cone is an equilateral triangle, as shown. The cross-section of the sphere is the incircle of the triangle and has radius 15 cm.
By the symmetry of the figure $\angle POR = 360° \div 6 = 60°$, and hence, triangle POR has angles 90°, 60° and 30° (so it is half of an equilateral triangle). Hence OR is twice OP and hence is 30 cm. Since $OT = OR$, it follows that PT is $15 + 30 = 45$ cm. This is the height of the cone.

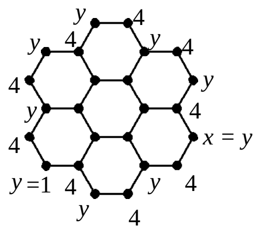


[*Alternatively: You may know that the medians of a triangle intersect at one third of their heights, so* $OP = 15$ cm *is one third of the height.*]

Solutions to the European Kangaroo Pink Paper

1. **D** $P = 6 + 12 + 20 = 38$, $Q = 4 + 9 + 16 = 29$ and $R = 2 + 6 + 12 = 20$ so $R < Q < P$.

2. **A** Because the sum of the numbers at the ends of each segment is always the same, the two vertices next to the 4 must be given the same number, say y. Then the sum of each edge on the lattice is $y + 4$. Every vertex adjacent to a y will be numbered 4; and every vertex adjacent to a 4 will be numbered y. This means there is an alternating sequence of 4, y, 4, y, … round the perimeter of the lattice. The vertex numbered 1 will be numbered y in this sequence, so $y = 1$. Continuing round to x, we see the vertex is numbered 1.



3. **C** The width of a rectangle with area 360 cm^2 and height 24 cm is $360 \div 24 = 15$ cm. This is the width of 5 tiles, so each tile is 3 cm wide, and since it is square, 3 cm high. The area of such a tile is 9 cm^2.

4. **D** Since the digits add up to 4, the first digit must be 1, 2, 3 or 4.

The only one starting with 4 is 4000.

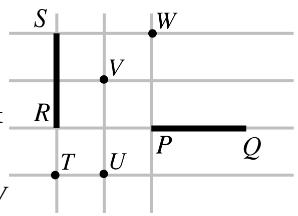
Those starting with 3 are 3100, 3010 and 3001.

Those starting with 2 have either two 2s or one 2 and two 1s: they are 2200, 2020, 2002, 2110, 2101, 2011. In descending order, they are 2200, 2110, 2101, 2020, 2011, 2002.

Those starting with 1 are all smaller than 2011, so can be ignored.

So 2011 is in 9th position.

5. **C** Label the horizontal line segment PQ, and the vertical line segment RS. A rotation of 90° anticlockwise about U, or 90° clockwise about W would map PQ onto RS.



When a rotation is performed, the distance of any point from the centre of rotation is preserved. Hence T cannot be a centre because it is a distance of 1 unit from R, but more than 1 unit from P and Q. Similarly V is less than 2 units from R and S, but more than 2 units from Q. So only U and W can be centres.

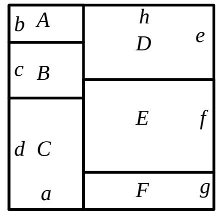
6. **C** The interior angles of a regular hexagon are all 120°. At any vertex of the hexagon, there are two squares and a triangle, so the angle of the triangle at that point must be $360° - 120° - 90° - 90° = 60°$. Hence the other two angles of the triangle must add to $180° - 60° = 120°$. The triangles are isosceles since they have two edges of length 1 unit, so the other two angles are equal and must be 60°. Therefore the triangles are equilateral.

Since the component shapes are all regular, every edge is 1 unit. The perimeter is then 12 units long.

12. **E** The average of all the marks is $\frac{96}{8} = 12$. We must therefore remove two marks that have an average of 12. The only two marks with this property are 10 and 14.

13. **B** Since Barcelona only let in one goal, the result of the game they lost must be 0-1 and the result of the game they drew must be 0-0. They scored three goals in total and hence the result of the game they won must be 3-0.

14. **D** Let the square have side x cm. Label each rectangle and its sides as shown where the units are cm. The perimeters are:



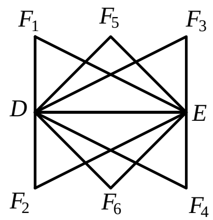
A	B	C
$2(a + b)$	$2(a + c)$	$2(a + d)$
D	E	F
$2(e + h)$	$2(f + h)$	$2(g + h)$

Thus the total of the perimeters is:

$$2(a + b + a + c + a + d + e + h + f + h + g + h) = 2\big(3(a + h) + (b + c + d) + (e + f + g)\big).$$

But $a + h = b + c + d = e + f + g = x$. Thus the perimeter is $2(3x + x + x) = 10x$. So $x = 12$ which means that the area of the paper is 144 cm^2.

15. **C** Since DE has length 2 cm, the point F must have a perpendicular distance of 1 cm from DE so that the area of triangle DEF is 1 cm^2. If the right angle is at D, then two points, F_1 and F_2, can be chosen, 1 cm directly above and below the point D. Similarly, if the right angle is at E, we obtain the points F_3 and F_4. Finally, DFE is a right angle when F lies on the circle whose diameter is DE. Since F must be 1 cm above or below the line DE, it must lie directly above or below the midpoint of DE. This gives F_5 and F_6 as shown. Lali can therefore draw 6 different points.



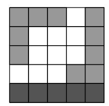
16. **B** Since a is a positive number less than 1 and b is greater than 1, then $a \times b < b$ and $a \div b < a < 1$. The value of $a + b$ is always greater than 1 and is also always greater than the value of b. Hence the largest value is $a + b$.

OR

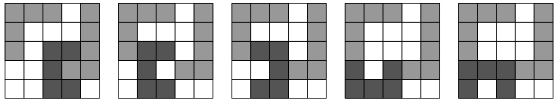
Since $0 < a < 1 < b$, then $a \div b < a < b < a + b$. Also $a \times b < b$. Hence $a + b$ has the largest value.

17. **E** To be divisible by 5, the last digit must be 0 or 5. To be divisible by 4, the last two digits must be a multiple of 4. There are no multiples of 4 with a units digit of 5 and hence $Y = 0$. So the five-digit number is $24X80$. To be divisible by 9, the sum of the digits must be a multiple of 9. The sum of the digits is $14 + X$. The smallest X can be is 0 and the largest X can be is 9. Therefore the sum of the digits is between 14 and 23. The only multiple of 9 in this range is 18 and therefore $X = 4$ and $X + Y = 4$.

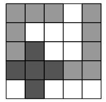
18. **D** By inspection, it is possible to spot the answer is shape D. We can justify this as follows: When Lina places any of the other shapes on the empty part of the board, the shape must cover at least one square on the bottom row so that Lina cannot place shape C on the board.



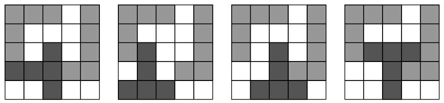
There are five ways that Lina can place shape A on the board, covering at least one square on the bottom row, as shown below. In the first diagram, she can then place shape E on the board. In the second and third diagrams, she can then place shape D on the board. In the final two diagrams, she can then place shapes D or E on the board.



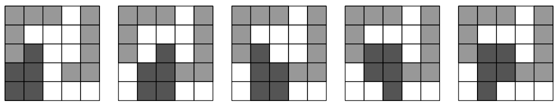
There is only one way that Lina can place shape B on the board, covering at least one square on the bottom row, as shown alongside. Lina can then place shapes D or E on the board.



There are four ways that Lina can place shape D on the board, covering at least one square on the bottom row, as shown below. In the first diagram, Lina can then place shape A on the board. In the second diagram, she can then place shape E on the board. In the third diagram, she can then place shape A on the board. In the fourth diagram, Lina cannot place any of the remaining shapes on the board. This is the shape she should choose.



Checking shape E shows that there are five ways it can be placed on the board, covering at least one square on the bottom row, as shown below. In the first two diagrams, Lina can then place shape A or D on the board. In the last three diagrams, she can then place shape D on the board.



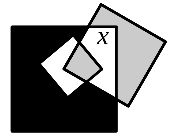
19. B Let the distance between Isaac and Max be IM, the distance between Max and Oscar be MO and the distance between Isaac and Oscar be IO. The three statements give: $IM > 2IO$; $MO > 2IM$; and $MO > 2IO$. Combining the first two inequalities gives $MO > 2IM = IM + IM > IM + 2IO > IM + IO$. However, MO, IM and IO are the three sides of a triangle so this contradicts the triangle inequality which states that for any three points, P, Q, R that $PQ \leqslant PR + QR$. Hence one of the first two statements is false.

Similarly, if we combine the last two inequalities, then $MO + MO > 2IM + 2IO$ and so $MO > IM + IO$; once again, this is not possible. So one of the last two statements is false. Since we know at least two statements are true, the middle statement is false and the first and third statements are true. Hence Max is lying.

20. D Let x be the number of times Myshko hits 5 and y be the number of times he hits each of 8 and 10.

Then $5x + 8y + 10y = 99$ which simplifies to $5x + 18y = 99$. The multiples of 18 less than 99 are 18, 36, 54, 72 and 90. The differences between these numbers and 99 are 81, 63, 45, 27 and 9 respectively. Of these, only 45 is a multiple of 5. Therefore $y = 3$ and $x = 9$. Myshko has hit the target $9 + 3 + 3 = 15$ times. Since he misses 25% of the time, Myshko had 20 shots in total.

21. D Let the area of the white hexagon be x cm^2, as indicated in the diagram.

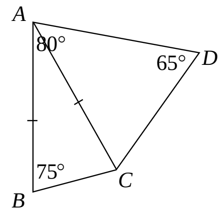


Then the black area is $49 - (9 + x) = (40 - x)\,\text{cm}^2$.

The total of the grey areas is $(25 - x)$ cm^2.

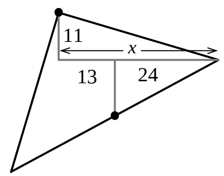
Thus the difference between the areas of the black and grey regions is $(40 - x) - (25 - x) = 15$ cm^2.

22. B Since $AB = AC$, triangle ABC is isosceles with $\angle BCA = \angle ABC = 75°$. So $\angle CAB = 180° - 2 \times 75° = 30°$. Since $\angle BAD = 80°$, $\angle CAD = 80° - 30° = 50°$. Now considering triangle ACD gives $\angle ACD = 180° - 50° - 65° = 65°$. Since $\angle ACD = \angle ADC = 65°$, triangle ACD is isosceles and $AC = AD$. Now $AB = AD$ and triangle ABD is also isosceles with $\angle ABD = \angle ADB = (180° - 80°) \div 2 = 50°$. Hence $\angle BDC = 65° - 50° = 15°$.

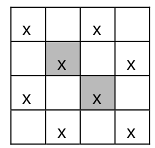


23. B Since the same letter stands for the same non-zero digit, the expression can be simplified to $\dfrac{K \times N \times A \times R \times O \times O}{M \times E}$. This expression will be smallest when the denominator is greatest and the numerator is smallest. Since M and E must be different digits and as large as possible, try $M \times E = 9 \times 8$. To minimize the expression, the numerator must be minimized but must also be divisible by $M \times E$. The denominator can be written as $2^3 \times 3^2$ so the numerator must also have these factors. Since the smallest possible value of the product of 5 different positive integers is $1 \times 2 \times 3 \times 4 \times 5 = 120$, the smallest possible value of our quotient is 2. Furthermore we require two multiples of 3 which suggests that we should take 3 and 6 as two of our numbers. To keep the others as small as possible, we are led to try 1, 2 and 4 and to take the repeated letter O to be 1. With K, N, A and R as 2, 3, 4 and 6 in any order, we obtain the minimum value 2.

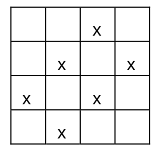
24. A The original shape constructed from two rectangles has base of length $11 + 13 = 24$. By considering the rearrangement, the lengths 11, 13, 24 and x can be identified as shown in the diagram. Hence $x = 13 + 24 = 37$.



25. C Since the two blue cells have a side in common, Mark could click alternate cells on the grid, as shown in the diagram. Mark will see one blue cell and he has made 8 clicks. If the blue cell is in one of the cells coloured grey in the diagram, he may need to click on four more cells to ensure that both blue cells appear. Mark has therefore made 12 clicks to ensure both blue cells appear.



The number of clicks can be reduced if Mark starts by clicking alternate cells except the two corner cells, as shown in the diagram. Mark has now made 6 clicks and, if any of these cells are blue, he will need at most 4 more clicks to ensure that both blue cells appear. Mark has made 10 clicks. If the initial 6 clicks do not show a blue cell, Mark then clicks on the two corner cells. One of these must be blue and he needs at most 2 more clicks to find the second blue cell. Mark has made 10 clicks.



The largest number of clicks Mark will need to make is 10.