

UKMT Junior Mentoring Scheme — Notes

A. EVEN, ODD AND PRIME NUMBERS

If a sum or difference of a pair of numbers is even, then both numbers are even or both are odd. To get an odd sum, one must be even and the other odd. If a product of two numbers is odd, both numbers must be odd.

That that 2 is a prime number. It is the only even prime number. All the other prime numbers are odd. Not every odd number is a prime number!

The number 1 is *not* a prime number. There would be problems if we included it, because every number is either prime or can be written as a product of a unique finite list of prime numbers. To consider 1 as a prime number would mean that we could keep on dividing out prime factors from a number *ad infinitum*.

B. USEFUL ALGEBRAIC FACTORISATIONS

You may be familiar with multiplying out brackets for squares:

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a - b)^2 = a^2 - 2ab + b^2.$$

A third result which is very useful is:

$$(a + b)(a - b) = a^2 - b^2,$$

which is sometimes referred to as the (factorisation of) the difference of two squares.

Since a square number is always non-negative, the second result leads to $a^2 - 2ab + b^2 \geq 0$.

This can be written in various ways, for example:

$$a^2 + b^2 \geq 2ab; \quad \frac{a^2 + b^2}{2} \geq ab; \quad \frac{a^2 + b^2}{2} \geq \sqrt{a^2b^2}.$$

If we now replace a^2 with x and b^2 with y , we deduce that

$$\frac{x + y}{2} \geq \sqrt{xy} \quad \text{for } x, y \geq 0.$$

Equality only occurs when $x = y$, and this is known as the *arithmetic mean–geometric mean inequality* (or “the AM–GM inequality” for short).

We can use this, for example, to verify that $\sqrt{15} < \frac{1}{2}(3 + 5)$.

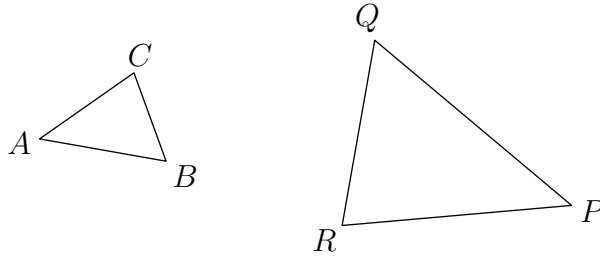
C. ADDING SERIES OF NUMBERS

Adding a set of numbers going up in regular steps, for example $3 + 8 + 13 + \dots + 98$, is easy with the right trick. Add 3 to 98, 8 to 93, 13 to 88 and so on, until you have gone all the way through to adding 98 to 3. You now have 20 pairs of numbers each adding to 101. (Why are there 20 pairs, and not $(98 - 3)/5 = 19$ pairs?) Adding all of these pairs together gives $20 \times 101 = 2020$, so the original series adds up to half of this, which is 1010. (Why do we have to halve our sum?)

A series of numbers going up in regular steps like this is called *arithmetic*.

D. SIMILAR AND CONGRUENT TRIANGLES

Two triangles are called *similar* if both have the same set of angles. The *corresponding* sides are then in a fixed ratio to each other. For example, consider these two triangles:



If $\angle ABC = \angle PQR$ and $\angle CAB = \angle RPQ$, so that by the sum of angles in a triangle, $\angle BCA = \angle QRP$, the triangle ABC and PQR are similar, so we have, for example,

$$\frac{AB}{PQ} = \frac{BC}{QR},$$

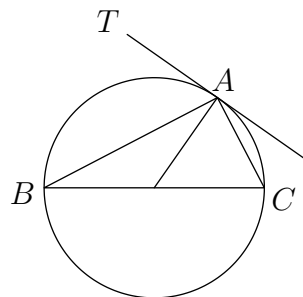
by comparing the sizes of the triangles, or

$$\frac{AB}{BC} = \frac{PQ}{QR},$$

which amounts to the same thing, but says that the ratio of corresponding pairs of sides is equal.

Triangles are called *congruent* if they have the same set of angles and also their corresponding sides are equal.

E. RIGHT ANGLES IN A CIRCLE

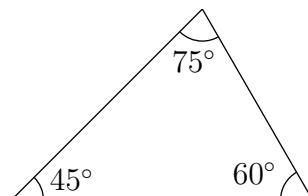


It is left to the reader to show that if O is the centre of the circle, then there are two isosceles triangles in the figure, which leads to the conclusion that $\angle BAC = 90^\circ$.

If AT is a tangent to the circle, then $\angle OAT = 90^\circ$.

F. CENTROID OF A TRIANGLE

Many solvers of geometry problems like to draw decent diagrams using geometrical instruments, redrawing them as some ideas become irrelevant and other ideas take on increasing importance. When trying to avoid a symmetrical or right-angled look, use angles near these:



Triangles have many interesting properties, including those of their “centres”. Here is just one of them:

There are three lines joining the vertices of the triangle to the mid-points of the opposite sides. These are called the *medians* of the triangle.

The three medians meet at the *centroid* (the “balancing point” of the triangle).

The centroid divides each median in the ratio 2 : 1.

Here is a proof of this result using areas of triangles.

Let $|ABC|$ stand for the area of triangle ABC . Let B' and C' be the midpoints of CA and AB respectively, and let the medians BB' and CC' meet at G . We draw the line AG and let AG meet BC at X . We are going to prove that X is the midpoint of BC . *Draw this diagram out for yourself to see what is happening.*

Using common bases or common heights, and using the fact that BB' is a median, it follows that

$$\begin{aligned} & |AB'B| = |CB'B| \text{ and } |AB'G| = |CB'G| \\ \therefore & |AGB| = |CGB| \\ \text{Similarly,} & |CGA| = |CGB| \text{ because } CC' \text{ is a median.} \\ \therefore & |AGB| = |AGC|. \end{aligned}$$

Now let $BX = k.CX$. By a similar argument, $|AGB| = k.|AGC|$.

Hence $k = 1$ and X is the midpoint of BC .

Furthermore, we have

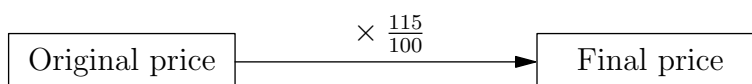
$$\begin{aligned} & |BGX| = |CGX| \\ \therefore & |BGX| = \frac{1}{2}|CGB| = \frac{1}{2}|ABG| \\ \therefore & GX = \frac{1}{2} \cdot AG. \end{aligned}$$

You are advised to measure the lengths in your diagram to check this works.

G. PERCENTAGE CHANGES

Regular polls and surveys have shown that these are one of the least understood arithmetical topics that have to be used in commercial activities by the general population. In the case of VALUE ADDED TAX,¹ a percentage is added by the store to the price they want to charge. VAT is currently 15%, but will be rising once again to 17.5% from January. Suppose they want to collect £12.99. They add 15% of this, namely £1.95, charge the customer £14.94 and pay the government the £1.95. Note that we can also understand this calculation as multiplying the original price that the shop wishes to charge by $\frac{115}{100}$ to determine the actual selling price.

A useful diagram for understanding this is as follows:



In the reverse case, suppose they advertise an article at a price of £19.99. To find the price that the store collects, we have to reverse the process and *divide* by $\frac{115}{100}$, giving £17.38, so that $£19.99 - £17.38 = £2.61$ goes to the government. Another way of thinking about this is that

¹A VAT registered trader claims back from the government the VAT on everything bought, but then pays the government the VAT on everything sold. In this way, the trader only pays the extra VAT on the markup made, i.e., the increase in value of the product. This is the reason for the name Value Added Tax.

out of every £115 charged to the customer, £100 goes to the shop-keeper and £15 goes to the government. Note that you *cannot* multiply by $\frac{85}{100}$ to find the amount the store-keeper gets.

There is also a term “percentage point”, and this is used to describe arithmetic differences between percentages. For example, if we think of an item as costing 115 percentage points in a shop, the shop-keeper gets 100 percentage points while the government gets 15. As another example, if the percentage of Year 6 boys gaining Level 4 or above in their Key Stage 2 SATs in 2002 was 35% and in 2009 it was 70%, we say that there has been a 35 percentage point improvement (being $70 - 35$), even though the actual percentage rise was 100%. This distinction causes much confusion, especially in the media.

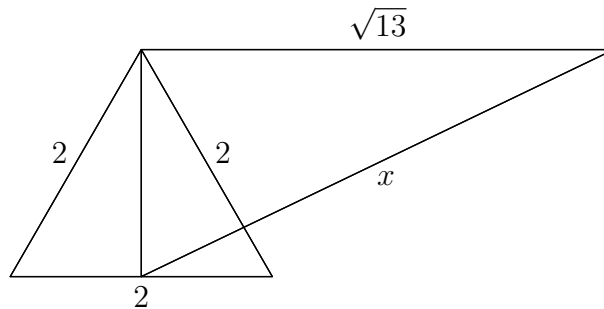
It is also wise to be suspicious of percentage statistics when applied to small numbers, and always question the base on which statistics are applied. Many users try to bamboozle their audience by quoting percentages which are, at best, irrelevant to the argument they are trying to make.

H. SURDS

Numbers like $\sqrt{5}$ are called *surd*s. Their appearance can be off-putting to say the least! However, the best way to treat them is like letters in algebra: they are what they do. In fact, it can help to replace a surd by a letter, as it may make the algebra less gruesome-looking.

Suppose you put c instead of $\sqrt{5}$. What happens when you find c^2 ? It comes to exactly 5, so you can put a number back instead of c^2 . What about c^3 ? That is $c^2 \times c = 5c = 5\sqrt{5}$. Note that $c^4 = c^2 \times c^2 = 5 \times 5 = 25$ exactly.

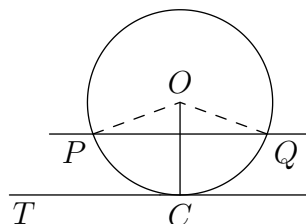
Consider an equilateral triangle with side length 2, as shown below. (Take the dimensions as shown and angles as right angles where they look like 90° in the diagram.) By Pythagoras’ Theorem applied to have the equilateral triangle, you can see the height of the triangle must be exactly $\sqrt{3}$. Applying Pythagoras’ Theorem again, you can then see that x must be *exactly* 4. Changing into decimals can never *prove* this result.



I. CIRCLE GEOMETRY

Why should a tangent be perpendicular to the radius of the circle at the point of contact?

We first show that a radius which bisects a chord is perpendicular to the chord. Consider this figure:



Let M be the midpoint of a chord PQ of the circle.

$OP = OQ$ as both are radii of the circle, so $\angle OPQ = \angle OMQ$.

Also, $PM = QM$, therefore $\triangle OPM$ is congruent to $\triangle OQM$, so $\angle OMP = \angle OMQ$.

Since $\angle OMP + \angle OMQ = 180^\circ$, it follows that $\angle OMP = \angle OMQ = 90^\circ$, so OM is perpendicular to PQ .

We can also prove the converse, namely, if OM is perpendicular to PQ , then M is the midpoint of PQ .

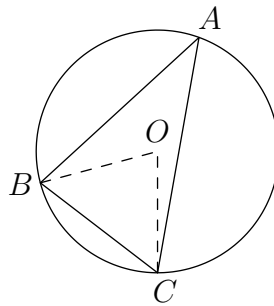
Now we can consider tangents and radii. Let OM meet the circle at C , as in the diagram above. Construct TC parallel to PQ and therefore perpendicular to OC . Take T be the point where OP meets the tangent TC . Since $OC > OM$, we must have $OT > OP$ as $\triangle OPM$ is similar to $\triangle OTC$. Therefore T must lie *outside* the circle.

This means that no point of the tangent TC can lie inside the circle, as we can draw a line from O to any point on the tangent, calling P the point where the line meets the circle and choosing Q on the circle such the PQ is parallel to CP .

Finally, as the tangent TC meets the circle at C , the tangent touches the circle at only this one point.

From radius, chord and tangent properties, we can deduce a number of significant theorems about angles in a circle:

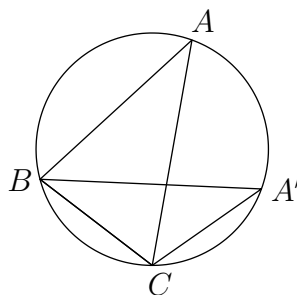
- (a) Angle at the centre is twice that at the circumference



$$\angle BOC = 2\angle BAC$$

This can be proven by drawing in the line OA and considering isosceles triangles.

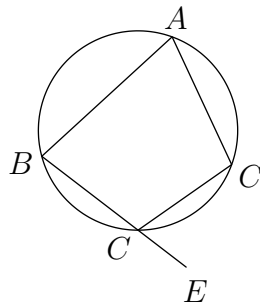
- (b) Angles on a chord in the same segment are equal



$$\angle BAC = \angle BA'C$$

This can be proven by using the previous theorem: if O is the centre of the circle, then $\angle BOC = 2\angle BAC = 2\angle BA'C$.

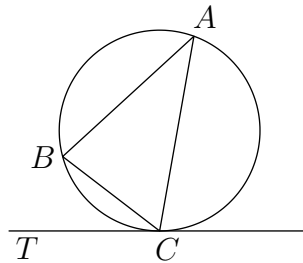
- (c) Opposite angles in a cyclic quadrilateral are supplementary



$$\begin{aligned}\angle DAB + \angle BCD &= 180^\circ \\ \therefore \angle DAB + \angle DCE &= 180^\circ.\end{aligned}$$

This can be proven either by using angle at the centre of a circle is twice the angle at the circumference. Alternatively, draw radii from the centre of the circle to the vertices of the quadrilateral, and notice that there are four isosceles triangles formed. Noting that the angles of a quadrilateral sum to 360° , the result then follows.

(d) The alternate segment theorem



$$\angle BCT = \angle BAC$$

This is easiest to show in the case that AC is perpendicular to the tangent, for then the angle $\angle ABC$ is a right angle (see note E above). If it is true in this case, then it is always true, as if B and C are fixed, the angles $\angle BAC$ and $\angle BCT$ are also fixed using the angles on a chord theorem.

For a more detailed account of geometry with many problems, consult *Plane Euclidean Geometry: Theory and Problems* by A.D. Gardiner and C.J. Bradley, published by the UK Mathematics Trust, <http://www.ukmt.org.uk/>.

J. SUBSCRIPT, INDEX OR SUFFIX NOTATION

If we have a sequence, we could name the terms of the sequence with letters in alphabetical order as a, b, c, \dots , but we would soon run out of letters. Instead, it is useful to use one letter for the whole sequence and to attach a small number next to it, written just below the line, called a *subscript*, *index* or *suffix*. This number tells us which position in the sequence we are at.

Here are some examples using the Fibonacci sequence, which is defined by adding two consecutive terms to yield the next term, starting with the two terms 1 and 1:

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \quad F_7 = 13, \quad F_8 = 21, \quad \text{etc.}$$

It is useful to take $F_0 = 0$, so that $F_0 + F_1 = F_2$.

- (i) F_8 is five positions along from F_3 .
- (ii) F_8^2 means the square of F_8 , and has the value $21^2 = 441$.

- (iii) F_n means a term in a general position n . Then F_{n-1} is the term before F_n , F_{n+1} is the term after F_n , and F_{n+2} is the term after F_{n+1} .
- (iv) F_{2n} means the $2n$ -th term, which is the term in the n th even position; F_{2n-1} is similarly the term in the n th odd position.
- (v) We can see that F_5^2 is one more than $F_4 \times F_6$. We normally leave out the times sign and write $F_5^2 = F_4 F_6 + 1$. Check that it is true that F_n^2 is one more or less than $F_{n-1} F_{n+1}$ for some other values of n .

K. THE GREEK ALPHABET

This is often used in mathematics when writers run out of (Latin) letters or wish to use a different style of letter. (There are occasionally times when mathematicians venture into Russian or Greek in addition, but these are fairly rare, thankfully!)

There are twenty-four characters in the Greek alphabet, usually written in this order:

A	α	alpha	a	I	ι	iota	i	P	ρ, ϱ	rho	r
B	β	beta	b	K	κ, \varkappa	kappa	k	Σ	σ, ς	sigma	s
Γ	γ	gamma	g	Λ	λ	lambda	l	T	τ	tau	t
Δ	δ	delta	d	M	μ	mu	m	Υ	υ	upsilon	u
E	ϵ, ε	epsilon	e	N	ν	nu	n	Φ	ϕ, φ	phi	f
Z	ζ	zeta	z	Ξ	ξ	xi	x	X	χ	chi	c
H	η	eta	h	O	o	omicron	o	Ψ	ψ	psi	y
Θ	θ, ϑ	theta	q	Π	π, ϖ	pi	p	Ω	ω	omega	w

In this table, the first column is the capital letter and the second column shows the lower case letters. (Seven letters are shown with common variant forms, and they are as follows: the two variants of the letters epsilon, theta and phi are both commonly found in mathematics books and papers; the second variant of the letters kappa, pi and rho are found, but are significantly rarer than the first form; finally, the variant form of sigma is very rare in mathematics books, and is used in Greek when a sigma appears at the end of a word.)

The third column is the name of the letter, with the bold-faced letter being the English pseudo-equivalents. While many pronounce ‘eta’ with the sound ‘ea’ as in ‘least’, classical scholars often use an ‘e’ sound as in ‘air’; the vowels in beta, zeta and theta are pronounced as in eta, and the ‘o’ in omega is as in ‘bold’.

The fourth column shows the keys which correspond to these letters in the standard keyboard mapping for the Symbol font on Windows.

L. RULES OF INEQUALITIES

The symbols $<$, \leq , $>$ and \geq are often mistrusted; many students often avoid them during problems, substituting $=$ instead, and then try to convince themselves later on that they have got the direction ($<$ or $>$) the right way round. There are some simple rules that will keep you on track.

As a general guideline, substitute in simple numbers as you go to check that you are on track.

The rules are as follows:

- (i) $a < b$ is the same as $b > a$.
For example, $3 < 5$ is equivalent to $5 > 3$.
- (ii) If $a < b$, then $a + x < b + x$ for any number x .
For example, since $3 < 5$, it follows that $-4 < -2$ by taking $x = -7$. The same is true for subtraction: if $a < b$, then $a - x < b - x$ for any number x .

- (iii) If $a < b$, then $ax < bx$ for any *positive* number x , but $ax > bx$ for any *negative* number x .
For example, since $3 < 5$, it follows that $12 < 20$ by taking $x = 4$, and $-12 > -20$ by taking $x = -4$.
- (iv) If $a < b$, then $-a > -b$.
For example, since $3 < 5$, it follows that $-3 > -5$. Note that this is equivalent to multiplying (or dividing) both sides by -1 , or by adding $-(a + b)$ to both sides (or subtracting $a + b$ from both sides).
- (v) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$ provided a and b both have the same sign (i.e., $ab > 0$).
For example, $3 < 5$ gives $\frac{1}{3} > \frac{1}{5}$ and $-5 < -3$ gives $-\frac{1}{5} < -\frac{1}{3}$. This result follows by dividing both sides by ab . Note that $-3 < 5$ but $-\frac{1}{3} \not> \frac{1}{5}$, as in this case we are dividing by $-3 \times 5 < 0$, so the inequality switches direction.

In summary, you may add or subtract to both sides, just as with an equation, and you may multiply or divide both sides by a positive number. But if you multiply or divide by a **negative** number, the symbol changes from $<$ to $>$ or vice versa.

There are some standard “gotchas” which are likely to catch you out. The most frequent cases are those which involve multiplying or dividing by numbers which are (or might be) negative:

- (a) Consider the case that $a < b$, and then the relation of xa to xb or $\frac{a}{x}$ to $\frac{b}{x}$. If we do not know the sign of x , the outcome will need to be split into two cases in your working, one with x positive, the other with x negative, and perhaps also the case $x = 0$.
- (b) Consider the situation $x < 3$. Is it necessarily true that $x^2 < 9$? Perhaps; if $x = 1$, then $x^2 < 9$, whereas if $x = -5$, then $x^2 > 9$.
- (c) Conversely, if $x^2 > 9$, then we might again have $x = -5$, so it does not follow that $x > 3$. This is very similar to the trap of saying: “ $x^2 = 9$ means $x = 3$.” We ignore the case $x = -3$ at our peril!!

To summarise, watch for hidden cases involving multiplying or dividing by negative numbers, taking squares or taking square roots.